Determinants and Pfaffians associated with D-Complete Posets (Topics in Young Diagrams and Representation Theory)

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Determinants and Pfaffians associated with D-Complete Posets

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Abstract

R.A. Proctor defined the d-complete posets and classified them into 15 irreducible ones. He showed that any d-complete poset is obtained by the slant sum of the irreducible ones. He also announced that he and Dale Peterson proved that every d-complete poset has hook length property. In this paper we give a combinatorial proof of the hook length property of the d-complete posets using the lattice path method. First we show that each generating function of (P,ω)-partitions is expressed as a determinant or a pfaffian for an irreducible d-complete poset P. Then we prove the determinant or pfaffian becomes a certain product for each irreducible P. We still don't finish all the 15 irreducible cases, but we found there appears several interesting determinats and Pfaffians. In this manuscript we give detailed proofs of some of them.

1 Introduction

In this manuscript we give some detailed versions of our proof which will appear in our forcecoming paper. First we tried to find proofs of the hook formulas of the so-called d-complete posets and we found there appears lots of interesting determinants and Pfaffians in the proof. Although those determinants and Pfaffians are themselves very interesting because they give certain variants of classical well-known determinants and Pfaffians, the calculations are rather direct and very long. In this manuscript we introduce detailed versions of some of them, and our proof in the forcecoming paper will be shotened vesion of them. One of the authors didn't have time to complete the proof of all of them this time, but the completed paper will appear in the near future. I would like to express sincere thanks to the another auther and H.Kawamuko for very fruitful discussions and suggestions.

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2 (P, ω)-Partitions

In [11], R.P.Stanley defined the (P, ω)-partitions and obtained the several results on their generating functions. In this section we introduce the notion of the (P, ω)-partitions and one variable generating functions of the (P, ω)-partitions for the d-complete posets P, which we desire to compute. By a *labeled* poset, we shall mean a pair (P, ω), where P is a finite poset and $\omega : P \to \mathbb{Z}_{>0}$ is an injective map that assigns labels to the elements of P where labels are positive integers. For covenience, we will often assume that $P = [n] = \{1, 2, \cdots, n\}$ as the base set and $\operatorname{Img} \omega = [n]$. One says that the labeling ω is *natural* if $x < y$ implies $\omega(x) < \omega(y)$ for all $x, y \in P$. The labeling dual to ω, denoted by ω^*, is defined by reversing the total order on $[n]$. Also the *order dual poset*, denoted by P^*, is defined by reversing the order on P, i.e. $x \leq y$ in P if and only if $x \geq y$ in P^*.

A *(P, ω)-partition* is a map $\sigma : P \to \mathbf{N}$ such that for all $x < y$ in P, we have

(i) $\sigma(x) \geq \sigma(y)$,

(ii) $\sigma(x) > \sigma(y)$ whenever $\omega(x) > \omega(y)$.

If ω is order-preserving, then σ is called for short a *P-partition*. If ω is order-reversing, then σ is called a *strict P-partition*. If $|\sigma| = \sum_{x \in P} \sigma(x) = m$, then σ is called a (P, ω)-partition of m and denoted by $\sigma \vdash m$. Let $\mathcal{A}(P, \omega)$ denote the set of all (P, ω)-partitions, and $\mathcal{A}(P)$ the set of all P-partitions.

Similarly we define a *reversed (P, ω)-partition* $\sigma : P \to \mathbf{N}$ by replacing the above conditions (i),(ii) by

(i') $\sigma(x) \leq \sigma(y)$,

(ii') $\sigma(x) < \sigma(y)$ whenever $\omega(x) > \omega(y)$.

And it is easy to see that the arguments are almost parallel. Let $\mathcal{R}(P, \omega)$ denote the set of all reversed (P, ω)-partitions. In this paper we only need the one variable generating function of (P, ω)-partitions weighted by $|\sigma|$:

$$F_{\mathcal{A}}(P, \omega; q) = \sum_{\sigma \in \mathcal{A}(P, \omega)} q^{|\sigma|}. \tag{1}$$

Similarly we also put

$$F_{\mathcal{R}}(P, \omega; q) = \sum_{\sigma \in \mathcal{R}(P, \omega)} q^{|\sigma|}. \tag{2}$$

The aim of this paper is to obtain the generating function for certain classes of finite posets and to show that it is expressed by a simple product formula. If $|P| = n$, then an order-preserving bijection $\tau : P \to \boldsymbol{n}$ is called a linear extension of P, where $\boldsymbol{n}$ denotes the n-elements chain. Let $\mathcal{L}(P)$ denote the set of linear extensions of P, and let $\mathcal{L}(P, \omega) = \{\omega \circ \tau^{-1} : \tau \in \mathcal{L}(P)\}$. Note that $\mathcal{L}(P^*) = \{\pi_0 \circ \tau : \tau \in \mathcal{L}(P)\}$ and $\mathcal{L}(P^*, \omega) = \{\omega \circ \tau^{-1} \circ \pi_0 : \tau \in \mathcal{L}(P)\}$, where P^* is the dual poset of P and π_0 is the longest element in S_n. Further we put $\mathcal{W}(P, \omega) = \{\tau \circ \omega^{-1} : \tau \in \mathcal{L}(P)\} \subseteq S_n$ and call its elements the *reading words* of the linear extensions relative to ω.

For every $\pi \in S_n$ let

$$D(\pi) = \{1 \leq i \leq n - 1 \, : \, \pi(i) > \pi(i + 1)\}$$

denote the descent set of π, and

$$A(\pi) = \{1 \le i \le n-1 \,:\, \pi(i) < \pi(i+1)\}$$

denote the ascent set of π. Further for $\pi \in S_n$ we let $\mathrm{maj}(\pi) = \sum_{i \in D(\pi)} i$ denote the *major index* of π and let $\min(\pi) = \sum_{i \in A(\pi)} i$ denote the *minor index* of π.

For any permutation $\pi \in S_n$ and $i \in [n]$ let

$$c_i(\pi) = \begin{cases} 0 & \text{if } i = 1, \\ c_{i-1}(\pi) + \delta(\pi^{-1}(i-1) > \pi^{-1}(i)) & \text{if } 2 \le i \le n. \end{cases}$$

where $\delta(*)$ equals 1 if $*$ is true, and 0 otherwise. Similarily we define

$$c'_i(\pi) = \begin{cases} 0 & \text{if } i = 1, \\ c'_{i-1}(\pi) + \delta(\pi^{-1}(i-1) < \pi^{-1}(i)) & \text{if } 2 \le i \le n. \end{cases}$$

We let $\mathrm{ch}(\pi) = \sum_{i=1}^{n} c_i(\pi)$ the *charge* of π, and let $\mathrm{coch}(\pi) = \sum_{i=1}^{n} c'_i(\pi)$ the *cocharge* of π. It is easy to see that $\mathrm{ch}(\pi) = \sum_{i \in D(\pi^{-1})}(n-i) = \min(\pi^{-1} \circ \pi_0)$ and $\mathrm{coch}(\pi) = \sum_{i \in A(\pi^{-1})}(n-i) = \mathrm{maj}(\pi^{-1} \circ \pi_0)$, where π_0 is the longest element in S_n. This implies $\mathrm{ch}(\pi) + \mathrm{coch}(\pi) = \binom{n}{2}$.

For any linear extension $\tau \in \mathcal{L}(P)$, let $D(\tau, \omega) = \{i \in [n-1] : \omega(\tau^{-1}(i)) > \omega(\tau^{-1}(i+1))\}$ denote the descent set of τ relative to ω, and we put

$$\mathcal{A}(P, \omega, \tau) = \left\{ \sigma \in \mathcal{A}(P, \omega) : \begin{array}{l} \sigma(\tau^{-1}(1)) \ge \cdots \ge \sigma(\tau^{-1}(n)) \text{ and} \\ i \in D(\tau, \omega) \Rightarrow \sigma(\tau^{-1}(i)) > \sigma(\tau^{-1}(i+1)) \end{array} \right\}$$

The fundamental theorem for (P, ω)-partition is

$$\mathcal{A}(P, \omega) = \bigcup_{\tau \in \mathcal{L}(P)} \mathcal{A}(P, \omega, \tau).$$

As a corollary of this theorem, we have

$$F_{\mathcal{A}}(P, \omega; q) = \frac{\sum_{\pi \in \mathcal{L}(P,\omega)} q^{\mathrm{maj}(\pi)}}{(q;q)_n} = \frac{\sum_{\pi \in \mathcal{W}(P,\omega)} q^{\mathrm{coch}(\pi_0 \circ \pi)}}{(q;q)_n}. \tag{3}$$

It is also easy to see that

$$F_{\mathcal{A}}(P, \omega^*; q) = \frac{\sum_{\pi \in \mathcal{L}(P,\omega)} q^{\min(\pi)}}{(q;q)_n} = \frac{\sum_{\pi \in \mathcal{W}(P,\omega)} q^{\mathrm{ch}(\pi_0 \circ \pi)}}{(q;q)_n}. \tag{4}$$

In [11], Stanley showed that

$$q^n F_{\mathcal{A}}(P, w^*; q) = (-1)^n F_{\mathcal{A}}\left(P, w; \frac{1}{q}\right).$$

Similarly we have

$$F_{\mathcal{R}}(P, \omega; q) = \frac{\sum_{\pi \in \mathcal{L}(P,\omega)} q^{\min(\pi \circ \pi_0)}}{(q;q)_n} = \frac{\sum_{\pi \in \mathcal{W}(P,\omega)} q^{\mathrm{ch}(\pi)}}{(q;q)_n},$$

and

$$F_{\mathcal{R}}(P, \omega^*; q) = \frac{\sum_{\pi \in \mathcal{L}(P,\omega)} q^{\mathrm{maj}(\pi \circ \pi_0)}}{(q;q)_n} = \frac{\sum_{\pi \in \mathcal{W}(P,\omega)} q^{\mathrm{coch}(\pi)}}{(q;q)_n},$$

for the generating functions of reversed (P,ω)-partitions. From now on we restrict our attention to the (P,ω)-partitions, and write $F(P,\omega;q)$ for $F_{\mathcal{A}}(P,\omega;q)$ for short as far as there is no fear of confusion.

In [10], Proctor defined the "slant sum" for d-complete posets. Here by abuse of terminology we use the word "slant sum" for any finite posets. Let P_1 be a finite poset and $y \in P_1$ be any element. Let P_2 be a finite connected poset which is non-adjacent to P_1 with the maximal elements $x_1, \cdots, x_m$. Then the *slant sum* of P_1 with P_2 at y, denoted by $P_1{}^y\backslash_{x_1,\cdots,x_m}P_2$, is the poset formed by creating the covering relations $x_1 \lessdot y, \cdots, x_m \lessdot y$.

Let P_1 be a finite poset and $z \in P_1$ be any element such that (z) is an n-element chain and z is covered by only one element $y \in P_1$. Here $(z) = \{w \in P : w \leq z\}$ is the principal order ideal generated by z. Let ω_1 be a labeling on P_1 whose restriction on (z) is a natural labeling and $\omega_1(y) > \omega_1(z)$. Let P_2 be any n-element connected poset which is non-adjacent to P_1 with the maximal elements $x_1, \cdots, x_m$, and let ω_2 be any labeling on P_2. Let P be the poset obtained by replacing the n-element chain (z) by the n-element poset P_2: i.e., $P = P_1'^y\backslash_{x_1,\cdots,x_m}P_2$, where P_1' is the poset obtained by removing the order ideal (z) from P_1 deleting the cover relation $y \gtrdot z$. Let M be an integer which is larger than any label appearing in ω_2. Define the labeling ω on P by $\omega|_{P_1'} = \omega_1 + M$ and $\omega|_{P_2} = \omega_2$, where $(\omega_1 + M)(w) = \omega_1(w) + M$ for $w \in P_1'$.

Lemma 2.1 *Then the generating function of (P,ω)-partitions is given by*

$$F(P,\omega;q) = (q;q)_n F(P_1,\omega_1;q)F(P_2,\omega_2;q)$$

Proof. The generating function of all (P_2,ω_2)-partitions σ such that $\sigma(x_1) \geq a, \cdots, \sigma(x_m) \geq a$, is $q^{na}F(P_2,\omega_2;q)$, while the generating function of all $(\boldsymbol{n},\omega_0)$-partition σ such that $\sigma(\hat{1}) \geq a$ is $\frac{q^{na}}{(q;q)_n}$, where $\boldsymbol{n}$ is the n-elements chain and ω_0 is the natural labeling. Thus there exists a function $f(q,a)$ such that

$$F(P_1,\omega_1;q) = \sum_{a=0}^{\infty} f(q,a)\frac{q^{na}}{(q;q)_n}.$$

Using this $f(q,a)$, the generating function $F(P,\omega;q)$ is expressed as

$$F(P,\omega;q) = \sum_{a=0}^{\infty} f(q,a)\ q^{na}F(P_2,\omega_2) = (q;q)_n F(P_1,\omega_1;q)F(P_2,\omega_2;q).$$

This proves the lemma. □

The above easy lemma is very fundamental to calculate the generating functions of varius posets. It assures that, if we prove a hook length property for a poset which contains a chain, then we can replace it by any poset which also has hook length property.

3 Admissible labelings

Let P be a finite poset with n elements and $\omega : P \to \boldsymbol{n}$ a labeling. For any elements $x, y \in P$ such that $x \lessdot y$, we define $\epsilon_\omega(x,y)$ by

$$\epsilon_\omega(x,y) = \begin{cases} 0 & \text{if } \omega(x) < \omega(y), \\ 1 & \text{if } \omega(x) > \omega(y). \end{cases}$$

For any interval $[a, b]$ of P, let $C(a, b)$ denote the set of all saturated chains $C = (x_0, x_1, \cdots, x_m)$ from a to b, i.e.,

$$a = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_m = b.$$

Thus, for any $C \in C(a, b)$ we define $\epsilon_\omega(C)$ by

$$\epsilon_\omega(C) = \sum_{i=1}^{m} \epsilon_\omega(x_{i-1}, x_i).$$

Definition 3.1 *A labeling ω of P is said to be admissible if it satisfies the followng condition (AL).*

(AL) For any maximal elements b_1 and b_2 of P, and for any element a of P which satisfies $a \leq b_1, b_2$,

$$\epsilon_\omega(C_1) = \epsilon_\omega(C_2)$$

holds for any $C_1 \in C(a, b_1)$ and $C_2 \in C(a, b_2)$.

Let $AL(P)$ denote the set of all admissible labelings of P.

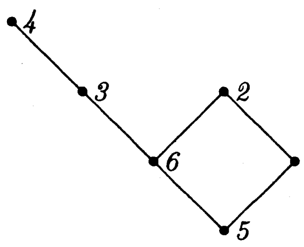
Note that, from the definition of the admissible labeling, it is clear that if $\omega \in AL(P)$ and a, b are elements of P such that $a \leq b$, then

$$\epsilon_\omega(C_1) = \epsilon_\omega(C_2)$$

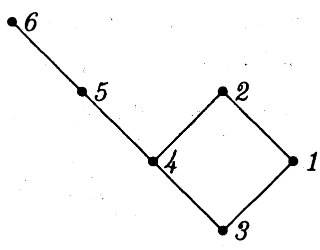
holds for any $C_1, C_2 \in C(a, b)$.

One also easily can see that any labeling of a tree is admissible since there exists a only one path from any element of P to the unique maximal element of P.

Example 3.2 *In the following poset the labeling ω_1 is admissible, but ω_2 is not.*



$(=: \omega_1)$



$(=: \omega_2)$

Let ω be an admissible labeling of a finite poset P. We define an order reversing map $\varphi_\omega : P \longrightarrow \mathbb{N}$ by

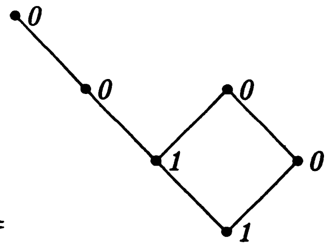
$$\varphi_\omega(x) := \begin{cases} 0 & \text{if } x \text{ is a maximal element in } P, \\ \varphi_\omega(y) & \text{if } x \lessdot y \text{ and } \omega(x) < \omega(y), \\ \varphi_\omega(y) + 1 & \text{if } x \lessdot y \text{ and } \omega(x) > \omega(y). \end{cases}$$

This implies that, in general, $\varphi_\omega(x)$ is defined by

$$\varphi_\omega(x) = \epsilon_\omega(C)$$

for a saturated chain $C \in C(x, y)$ and a maximal element y in P. It is easy to see, from the definition of the admissible labeling, that φ_ω is well-defined and become a (P, ω)-partition.

Example 3.3 *The following φ_{ω_1} corresponds to ω_1 in Ex. 3.2 and* $|\varphi_{\omega_1}| = 2$

$\varphi_{\omega_1} =$ 

The following theorem is the main result in this section.

Theorem 3.4 *Let ω be an admissible labeling of a finite poset P. Then we have*

$$\sum_{\varphi \in \mathcal{A}(P,\omega)} q^{|\varphi|} = q^{|\varphi_\omega|} \sum_{\varphi \in \mathcal{A}(P)} q^{|\varphi|}.$$

Proof. Recall that $\mathcal{A}(P,\omega)$ denote the set of all (P,ω)-partitions and $\mathcal{A}(P)$ the set of all P-partitions. Define $\Phi : \mathcal{A}(P) \to \mathcal{A}(P,\omega)$ by

$$\Phi(\varphi)(x) = \varphi(x) + \varphi_\omega(x).$$

If we show that this gives a bijection between $\mathcal{A}(P)$ and $\mathcal{A}(P,\omega)$, then the desired identity holds since $|\Phi(\varphi)| = |\varphi_\omega| + |\varphi|$. In fact, if we define $\Phi' : \mathcal{A}(P,\omega) \to \mathcal{A}(P)$ by

$$\Phi'(\varphi')(x) = \varphi'(x) - \varphi_\omega(x),$$

then it is easily checked that Φ and Φ' are well-defined. From the definition it is clear that Φ and Φ' are inverse maps of each other and this proves our theorem. □

Conjecture 3.5 *Let P be a finite poset and ω a labeling of P. Then the following two conditions are equivalent.*

(i) ω is an admissible labeling.

(ii) There exists $m \in \mathbb{N}$ such that

$$\sum_{\varphi \in \mathcal{A}(P,\omega)} q^{|\varphi|} = q^m \sum_{\varphi \in \mathcal{A}(P)} q^{|\varphi|}.$$

The condition (ii) can be replaced by the following condition (ii)'.
(ii)' There exists $m \in \mathbb{N}$ and a linear extension ω_0 such that

$$\sum_{\pi \in \mathcal{W}(P,\omega)} q^{\mathrm{coch}(\pi_0 \circ \pi)} = q^m \sum_{\pi \in \mathcal{W}(P,\omega_0)} q^{\mathrm{coch}(\pi_0 \circ \pi)}.$$

4 The Lattice Path Method

In this section we present some lemmas which will be needed to obtain the generating function of the posets which will appear in the following sections.

Let $\lambda = (\lambda_1, \cdots, \lambda_r)$ be a partition. i.e. $\lambda_1 \geq \cdots \geq \lambda_r > 0$. Let $Q = \{(i,j) : i,j \in \mathbb{P}\}$ denote the set of integral points in the strict fourth quadrant of the plane. We define the order in Q by the relation $(i_1, j_1) \leq (i_2, j_2)$ if and only if $i_1 \geq i_2$ and $j_1 \geq j_2$. Let $P = D(\lambda) =$

$\{(i,j) : 1 \le i \le r, 1 \le j \le \lambda_i\}$ be the filter of Q, and consider it as a finite poset.

In this paper we consider two different labelings of P. One is called *column-strict* labeling, which is defined by

$$\omega_c(i,j) = \sum_{k=1}^{i-1} \lambda_k + \lambda_i + 1 - j,$$

and the other is called *row-strict* labeling defined by

$$\omega_r(i,j) = \sum_{k=i+1}^{r} \lambda_k + j.$$

Alternatively, when ω is column-strict (resp. row-strict) labeling, a (P,ω)-partition is called column-strict (resp. row-strict) *reverse plane partition* which is defined to be a filling of Young diagram λ by nonnegative integers:

$$\begin{array}{cccccc} a_{11} & a_{12} & \cdots & & \cdots & a_{1\lambda_1} \\ a_{21} & a_{22} & \cdots & a_{2\lambda_2} & & \\ \vdots & \ddots & \mathinner{\mkern2mu\raise1pt\hbox{.}\mkern2mu\raise4pt\hbox{.}\mkern2mu\raise7pt\hbox{.}\mkern1mu} & & & \\ a_{r1} & \cdots & a_{r\lambda_r} & & & \end{array}$$

which satisfies the conditions:

(i) the entries increase weakly (resp. strongly) from left to right along each row,

(ii) the entries increase strongly (resp. weakly) from top to bottom along each column.

Let O denote the "octant" subposet of Q formed by taking the weakly upper triangular portion of Q: $O = \{(i,j) \in Q : j \ge i\}$. Let $\mu = (\mu_1, \cdots, \mu_r)$ be a strict partition, i.e. $\mu_1 > \cdots > \mu_r > 0$. Let $P = D(\mu) = \{(i,j) : 1 \le i \le r, i \le j \le i - 1 + \mu_i\}$ be the filter of O. Similarly we define the column-strict (resp. row-strict) labeling ω_c (resp. ω_r) on P by

$$\omega_c(i, i-1+j) = \sum_{k=1}^{i-1} \mu_k + \mu_i + 1 - j \quad \left(\text{resp. } \omega_r(i, i-1+j) = \sum_{k=i+1}^{r} \mu_k + j\right).$$

By the similar argument as above, when ω is column-strict (resp. row-strict) labeling, a (P,ω)-partition φ is identified with a column-strict (resp. row-strict) *shifted reverse plane partition* which is defined to be a filling of shifted Young diagram μ by nonnegative integers:

$$\begin{array}{cccccc} a_{11} & a_{12} & \cdots & \cdots & & a_{1\mu_1} \\ & a_{22} & \cdots & \cdots & & a_{2,1+\mu_2} \\ & & \ddots & & & \vdots \\ & & & & a_{rr} & a_{r,r-1+\mu_r} \end{array}$$

which satisfies the same conditions as above. The strict partition μ is called the *shape* of φ and the entries in the main diagonal $(a_{11}, \cdots, a_{rr})$ form a strict reverse partition called the *profile* of φ.

Lemma 4.1 *Let $\mu = (\mu_1 > \cdots > \mu_r > 0)$ be a strict partition and let $a = (0 \le a_1 < \cdots < a_r)$ be a strict reverse partition.*

(1) Then the generating function of column-strict shifted reverse plane partitions of shape μ and profile a is given by

$$\frac{1}{\prod_{i=1}^{r}(q;q)_{\mu_i-1}} \det\left(q^{\mu_i a_j}\right)_{1\le i,j\le r} \tag{5}$$

(2) The generating function of row-strict shifted reverse plane partitions of shape μ and profile a is given by

$$\frac{q^{\;{}_{i=1}^{r}\binom{\mu_i}{2}}}{\prod_{i=1}^{r}(q;q)_{\mu_i-1}} \det\left(q^{\mu_i a_j}\right)_{1\le i,j\le r} \tag{6}$$

Let $0 \le r \le n \le N$ be nonnegative integers. Let B be an arbitrary N by N skew-symmetric matrix; that is, $B = (b_{ij})$ satisfies $b_{ij} = -b_{ji}$. Let $T = (t_{ik})_{1\le i\le n, 1\le k\le N}$ be any n by N matrix, For a row index set $I = \{i_1, \cdots, i_r\}$ and a column index set $J = \{j_1, \cdots, j_r\}$, let T_J^I denote the submatrix obtained by choosing the rows indexed by I and the columns indexed by J. Especially, in the case of $I = [n]$, we write T_J for T_J^I. We cite a useful theorem from [6], which expresses a sum of minors by one Pfaffian.

Theorem 4.2 *Let $n \le N$ and assume n is even. Let $T = (t_{ik})_{1\le i\le n, 1\le k\le N}$ be any n by N matrix, and let $B = (b_{ik})_{1\le i,k\le N}$ be any N by N skew symmetric matrix. Then*

$$\sum_{\substack{I\subseteq[N]\\ \sharp I=n}} \operatorname{pf}(B_I^I)\det(T_I) = \operatorname{pf}(Q), \tag{7}$$

where Q is the n by n skew-symmetric matrix defined by $Q = TB\,{}^tT$, i.e.

$$Q_{ij} = \sum_{1\le k<l\le N} b_{kl}\det(T_{kl}^{ij}), \qquad (1 \le i, j \le n). \tag{8}$$

As a corollary of this theorem and the above lemma, we obtain the following lemma.

Lemma 4.3 *Let r, s be integers such that $0 \le s \le r$ and $r + s$ is even. Let $\mu = (\mu_1 > \cdots > \mu_r > 0)$ be a strict partition and let $a = (0 \le a_1 < \cdots < a_s)$ be a strict reverse partition. Then the generating function of column-strict (resp. row-strict) shifted reverse plane partitions such that its shape is μ and the first s parts of its profile is equal to a is given by*

$$\operatorname{pf}\left(\begin{array}{c|c} \frac{q^{(\mu_i+\mu_j)(a_s+1)}}{(q;q)_{\mu_i}(q;q)_{\mu_j}}\frac{q^{\mu_i}-q^{\mu_j}}{(\mu_i+\mu_j)_q} & \frac{q^{\mu_i a_{r+s+1-j}}}{(q;q)_{\mu_i-1}} \\ \hline -\frac{q^{\mu_j a_{r+s+1-i}}}{(q;q)_{\mu_j-1}} & 0 \end{array}\right).$$

Proof. This theorem is obtained from the definition of shifted plane partions and the lattice path method. For details, see [6].

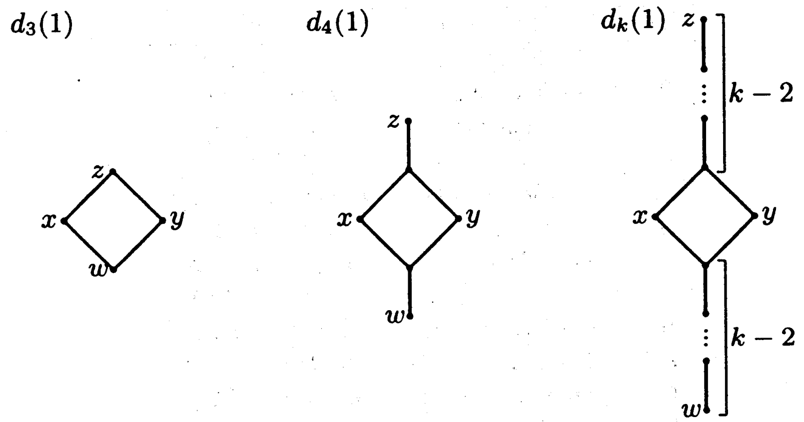
5 d-Complete Posets

In this section we briefly recall the basic definitions and properties of the d-complete posets. The reader should refer to [10] for details.

Let P be a finite poset. If $x, y \in P$, then we say y covers x if $x < y$ and no $z \in P$ satisfies $x < z < y$. When y covers x, we denote $y \gtrdot x$. An *order ideal* of P is a subset I of P such that if $x \in I$ and $y \le x$, then

$y \in I$. Similarly a *filter* is a subset F of P such that if $x \in F$ and $y \geq x$, then $y \in F$. The order ideal $\langle x \rangle$ is the *principal order ideal* generated by x.

For $k \geq 3$, the *double-tailed diamond* poset $d_k(1)$ has $2k-2$ elements, of which two are incomparable elements in the middle rank and $k-2$ apiece form chains above and below the two incomparable elements. The $k-2$ elements above the two incomparable elements are called *neck* elements. For $k \geq 3$, we say that an interval $[w, z]$ is a *d_k-interval* if it is isomorphic to $d_k(1)$. Further, for $k \geq 4$, we say that an interval $[w, z]$ is a *d_k^--interval* if it is isomorphic to $d_k(1) \setminus \{t\}$, where t is the maximal element of $d_k(1)$. A subposet $\{w, x, y, z\}$ of P is a *diamond* if z covers x and y, and each of x and y cover w. The following figures shows how the d_k-interval looks like.



A poset P is d_3-complete if it satisfies the following conditions:

(1) Whenever two elements x and y cover a third element w, there exists a fourth element z which covers both x and y,

(2) If $\{w, x, y, z\}$ is a diamond in P, then z covers only x and y in P, and

(3) No two elements x and y can cover each of two other elements w and w'.

Let $k \geq 4$. Suppose $[w, y]$ is a d_k^--interval in which x is the unique element covering w. If there is no $z \in P$ covering y such that $[w, z]$ is a d_k-interval, then $[w, y]$ is a *incomplete* d_k^--interval. If there exists $w' \neq w$ which is covered by x such that $[w', y]$ is also a d_k^--interval, then we say that $[w, y]$ and $[w', y]$ overlap. For any $k \geq 4$, a poset P is *d_k-complete* if:

(1) There are no incomplete d_k^--intervals,

(2) If $[w, z]$ is a d_k-interval, then z covers only one element in P, and

(3) There are no overlapping d_k^--intervals.

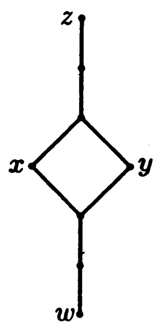
Definition 5.1 A poset P is *d-complete* if it is d_k-complete for every $k \geq 3$.

It is an easy consequence of the definition that, if P is d-complete and connected, then it has a unique maximum element $\hat{1}$ and every saturated chain from an element w to $\hat{1}$ has the same length (See [10]). A *top tree element* $x \in P$ is an element such that every element $y \geq x$ is covered by at most one other element. The *top tree* T of P consists of all top tree elements. An element $y \in P$ is *acyclic* if $y \in T$ and it is not in the neck of any d_k-interval for any $k \geq 3$. An element is *cyclic* if it is not acyclic. Let P_1 be a d-complete poset containing an acyclic element y and let P_2 be a connected d-complete poset which shares no element with P_1. It is known that P_2 has the unique maximal element which is denoted by x. Then the *slunt sum* of P_1 with P_2, denoted by $P_1{}^x\backslash_y P_2$, is the poset formed by creating a new covering relation $x > y$. A *d*-complete poset P is said to be *slant irreducible* if it is connected and it cannot be written as a slant sum of two non-empty *d*-complete posets. A slant irreducible poset which has two or more elements is called an irreducible components. Proctor[10] showed that, if P is connected *d*-complete poset, it is uniquely decomposed into a slant sum of one element posets and irreducible components. He also classified the irreducible components and showed that 15 disjoint classes of irreducible components $C_1, \ldots, C_{15}$ in the following table exhaust the set of all irreducible components.

Class	Colloquially	Name
1	Shapes	$a_n[0; g, h; \lambda]$
2	Shifted Shapes	$d_n[1; 1, h; \mu]$
3	Birds	$y_n[f; g, h]$
4	Insets	$e_n[f; 1, h; 4; \lambda]$
5	Tailed Insets	$a_n[f; 1, h; 5; \lambda, \mu]$
6	Banners	$e_n[f; 1, h; 6; \lambda]$
7	Nooks	$e_n[f; 1, h; 7; \lambda]$
8	Swivels	$e_n[f; 1, 2; 8; \lambda]$
9	Tailed Swivels	$e_n[f; 1, 2; 9; \lambda, \mu]$
10	Tagged Swivels	$e_n[f; 1, 2; 10; \lambda]$
11	Swivel Shifteds	$e_n[f; 1, 2; 11; \mu]$
12	Pumps	$e_n[f; 1, 2; 12; \lambda]$
13	Tailed Pumps	$e_n[f; 1, 2; 13; \lambda]$
14	Near Bats	$e_n[f; 1, 2; 14]$
15	Bat	$e_7[f; 1, 2; 15]$

The reader can find the pictures of these posets in the next section.

Definition 5.2 *Let P be a d-complete poset. For any element $z \in P$ we define its hook length, denoted by $h(z)$ as follows. If z is not in the neck of any d_k-interval, then $h(z)$ is the number of elements of the pricipal order ideal generated by z, i.e. $h(z) = \sharp(z)$. If z is included in the neck of some d_k-intervel, then, from the definition of the d-complete posets, we can take the unique element $w \in P$ such that $[w, z]$ is d_l-interval for some $l \leq k$. Let x and y be the two incomparable elements in this d_l-interval. Then we define the hook length $h(z)$ recursively by $h(z) = h(x) + h(y) - h(w)$.*



The aim of this paper is to prove the Frame-Robinson-Thrall type hook formula for d-complete posets, which says the number of linear extensions of an n-elements d-complete poset P is equal to $\frac{n!}{\prod_{x \in P} h(x)}$ and its q-analogue, which reads $\sum_{\pi \in \mathcal{W}(P,\omega)} q^{\mathrm{coch}(\pi_0 \circ \pi)} = q^{n(P,\omega)} \frac{(q;q)_n}{_{x \in P} 1-q^{h(x)}}$ where ω is some labeling and $n(P,\omega)$ is some constant determined by (P,ω). First we want to prove this q-hook formulas for the 15 classes of irreducible components (in fact we consider so-called extended irreducible components P in which a chain is attached to each acyclic element of each irreducible component) from which we can deduce q-hook formulas for any d-complete posets by Lemma 2.1. For each irreducible component P, we first calculate $\sum_{\pi \in \mathcal{W}(P,\omega)} q^{\mathrm{coch}(\pi_0 \circ \pi)}$ from the generating function $F(P,\omega;q)$ of (P,ω)-partitions for an appropriate labeling ω of P by the equation (3). Then we make the generating function into a product form, which is equivalent to $q^{n(P,\omega)} \frac{(q;q)_n}{_{x \in P}(1-q^{h(x)})}$. At this point we saw the generating function equals the product form for 13 classes of the irreducible components but still 2 classes remains unsolved. The concrete form of the fook formula in the product form for each irreducible component will be found in the next section.

6 Proof of Hook Formulas

In this paper, when we say a hook-formula, it always means a q-hook formula.

To begin with, we sum up some useful identities to be used in the following subsections. Here x, y, a, b, c denote arbitrary integers. First an easy direct calculation shows the following two identities.

$$q^y(x+a)_q(x+b)_q - q^x(y+a)_q(y+b)_q = (q^y - q^x)(x+y+a+b)_q \tag{9}$$

$$\begin{aligned} &q^y(x+2y+a+b+c)_q(x+a)_q(x+b)_q(x+c)_q \\ &\quad - q^x(2x+y+a+b+c)_q(y+a)_q(y+b)_q(y+c)_q \\ &\quad = (q^y - q^x)_q(x+y+b+c)_q(x+y+c+a)_q(x+y+a+b)_q \end{aligned} \tag{10}$$

Further we enumerate several determinant formulas which are immediate consequences of simple calculations and the above formulas.

$$\begin{vmatrix} 1 & \frac{1}{(x)_q} \\ 1 & \frac{1}{(y)_q} \end{vmatrix} = \frac{q^y - q^x}{(x)_q(y)_q} \tag{11}$$

$$\begin{vmatrix} 1 & \frac{1}{(y+a)_q} \\ 1 & \frac{1}{(y+b)_q} \end{vmatrix} - \begin{vmatrix} 1 & \frac{1}{(x+a)_q} \\ 1 & \frac{1}{(x+b)_q} \end{vmatrix} = \frac{(q^b - q^a)(q^y - q^x)(x+y+a+b)_q}{(x+a)_q(x+b)_q(y+a)_q(y+b)_q} \tag{12}$$

$$\begin{vmatrix} \frac{1}{(a-x)_q} & 1 & \frac{q^x}{(x+b)_q(x+c)_q} \\ \frac{1}{(a-y)_q} & 1 & \frac{q^y}{(y+b)_q(y+c)_q} \\ \frac{1}{(a-z)_q} & 1 & \frac{q^z}{(z+b)_q(z+c)_q} \end{vmatrix} = \frac{q^{-z}(q^z - q^x)(q^z - q^y)}{(a-z)_q(z+b)_q(z+c)_q} \begin{vmatrix} \frac{q^{a-x}}{(a-x)_q} & \frac{(x+z+b+c)_q}{(x+b)_q(x+c)_q} \\ \frac{q^{a-y}}{(a-y)_q} & \frac{(y+z+b+c)_q}{(y+b)_q(y+c)_q} \end{vmatrix} \tag{13}$$

$$\begin{aligned} &\begin{vmatrix} \frac{q^{-x}}{(a-x)_q(b-x)_q} & 1 & \frac{q^x}{(x+c)_q(x+d)_q} \\ \frac{q^{-y}}{(a-y)_q(b-y)_q} & 1 & \frac{q^y}{(y+c)_q(y+d)_q} \\ \frac{q^{-z}}{(a-z)_q(b-z)_q} & 1 & \frac{q^z}{(z+c)_q(z+d)_q} \end{vmatrix} \\ &= \frac{q^{-z}(q^z - q^x)(q^z - q^y)}{(a-z)_q(b-z)_q(z+c)_q(z+d)_q} \begin{vmatrix} \frac{q^{-x}(a+b-x-z)_q}{(a-x)_q(b-x)_q} & \frac{(x+z+c+d)_q}{(x+c)_q(x+d)_q} \\ \frac{q^{-y}(a+b-y-z)_q}{(a-y)_q(b-y)_q} & \frac{(y+z+c+d)_q}{(y+c)_q(y+d)_q} \end{vmatrix} \end{aligned} \tag{14}$$

As coloraries of (13) and (14), we obtain the following determinants.

$$\begin{vmatrix} \frac{1}{(a-x)_q} & 1 & \begin{vmatrix} 1 & \frac{1}{(x+b)_q} \\ 1 & \frac{1}{(x+c)_q} \end{vmatrix} \\ \frac{1}{(a-y)_q} & 1 & \begin{vmatrix} 1 & \frac{1}{(y+b)_q} \\ 1 & \frac{1}{(y+c)_q} \end{vmatrix} \\ \frac{1}{(a-z)_q} & 1 & \begin{vmatrix} 1 & \frac{1}{(z+b)_q} \\ 1 & \frac{1}{(z+c)_q} \end{vmatrix} \end{vmatrix} = \frac{q^{-z}(q^z-q^x)(q^z-q^y)(q^c-q^b)}{(a-z)_q(z+b)_q(z+c)_q} \begin{vmatrix} \frac{q^{a-x}}{(a-x)_q} & \frac{(x+}{(x+} \\ \frac{q^{a-y}}{(a-y)_q} & \frac{(y+}{(y+} \end{vmatrix} \tag{15}$$

$$\begin{vmatrix} \begin{vmatrix} 1 & \frac{1}{(a-x)_q} \\ 1 & \frac{1}{(b-x)_q} \end{vmatrix} & 1 & \begin{vmatrix} 1 & \frac{1}{(x+c)_q} \\ 1 & \frac{1}{(x+d)_q} \end{vmatrix} \\ \begin{vmatrix} 1 & \frac{1}{(a-y)_q} \\ 1 & \frac{1}{(b-y)_q} \end{vmatrix} & 1 & \begin{vmatrix} 1 & \frac{1}{(y+c)_q} \\ 1 & \frac{1}{(y+d)_q} \end{vmatrix} \\ \begin{vmatrix} 1 & \frac{1}{(a-z)_q} \\ 1 & \frac{1}{(b-z)_q} \end{vmatrix} & 1 & \begin{vmatrix} 1 & \frac{1}{(z+c)_q} \\ 1 & \frac{1}{(z+d)_q} \end{vmatrix} \end{vmatrix}$$

$$= \frac{q^{-z}(q^z-q^x)(q^z-q^y)(q^b-q^a)(q^d-q^c)}{(a-z)_q(b-z)_q(z+c)_q(z+d)_q} \begin{vmatrix} \frac{q^{-x}(a+b-x-z)_q}{(a-x)_q(b-x)_q} & \frac{(x+z+c+d)_q}{(x+c)_q(x+d)_q} \\ \frac{q^{-y}(a+b-y-z)_q}{(a-y)_q(b-y)_q} & \frac{(y+z+c+d)_q}{(y+c)_q(y+d)_q} \end{vmatrix}. \tag{16}$$

Further the following identities are also frequently used in what follows.

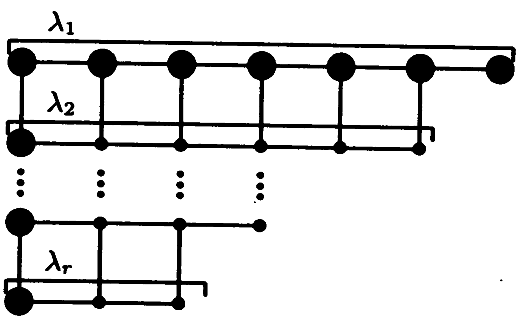
$$\sum_{x=0}^{\infty} \begin{bmatrix} x+a \\ a \end{bmatrix} q^{bx} = \frac{(q;q)_{b-1}}{(q;q)_{a+b}} \tag{17}$$

$$\sum_{x=a}^{\infty} \begin{bmatrix} x \\ a \end{bmatrix} q^{bx} = q^{ab}\frac{(q;q)_{b-1}}{(q;q)_{a+b}} \tag{18}$$

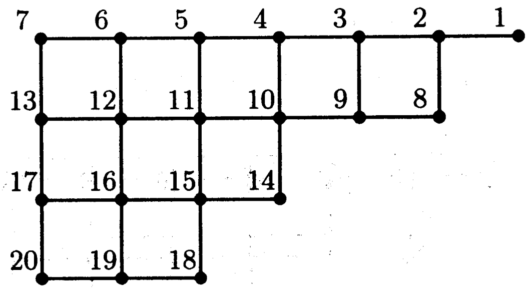
6.1 Shapes

First of all, the hook-length property of shapes is a well-known classical formula (see [3]), but here we briefly review how to obtain the generating function $F(P,\omega;q)$. For a detailed explanation of hook formulas for shapes and shifted shapes, see [5].

Shapes $(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 1)$



We put a labeling ω as follows which we call the column-strict labeling. Then a (P,ω)-partition is usually called a column-strict tableau.



Theorem 6.1 *If ω is the column-strict labeling of a shape $\lambda = (\lambda_1, \cdots, \lambda_r)$, then*

$$F(P,\omega;q) = q^{n(\lambda)} \frac{\prod_{1\le i<j\le r} (\lambda_i - \lambda_j - i + j)_q}{\prod_{i=1}^{r} (q;q)_{\lambda_i + r - i}}.$$

Here $n(\lambda) = \sum_{i=1}^{r} (i-1)\lambda_i$.

Proof. Since a (P,ω)-partition is a column-strict tableau, we have

$$F(P,\omega;q) = s_\lambda(1,q,q^2,q^3,\cdots),$$

where $s_\lambda(x_1,x_2,\cdots)$ is the Schur function with infinitely many variables. Assume $n \ge r = \ell(\lambda)$. From the definition of the Shur functions

$$s_\lambda(x_1,\cdots,x_n) = \frac{\det\left(x_i^{\lambda_j+n-j}\right)_{1\le i,j\le n}}{\det\left(x_i^{n-j}\right)_{1\le i,j\le n}},$$

and the Vandermonde determinant, $s_\lambda(1,q,\cdots,q^{n-1})$ equals

$$\frac{\det\left(q^{(i-1)(\lambda_j+n-j)}\right)_{1\le i,j\le n}}{\det\left(q^{(i-1)(n-j)}\right)_{1\le i,j\le n}} = \frac{\prod_{1\le i<j\le n}(q^{\lambda_j+n-j} - q^{\lambda_i+n-i})}{\prod_{1\le i<j\le n}(q^{n-j} - q^{n-i})}$$

If we put $n \to \infty$, then we obtain the theorem. □

Corollary 6.2 *If $\lambda = (\lambda_1,\cdots,\lambda_r)$ is a partition, then*

$$\det\left(\frac{1}{(q;q)_{\lambda_i-i+j}}\right)_{1\le i,j\le r} = q^{n(\lambda)} \frac{\prod_{1\le i<j\le r} (\lambda_i - \lambda_j - i + j)_q}{\prod_{i=1}^{r} (q;q)_{\lambda_i+r-i}} \tag{19}$$

$$\det\left(\frac{q^{\binom{\lambda_i-i+j}{2}}}{(q;q)_{\lambda_i-i+j}}\right)_{1\le i,j\le r} = q^{\sum_{i=1}^{r}\binom{\lambda_i}{2}} \frac{\prod_{1\le i<j\le r} (\lambda_i - \lambda_j - i + j)_q}{\prod_{i=1}^{r} (q;q)_{\lambda_i+r-i}} \tag{20}$$

Here we use the convention $\frac{1}{(q;q)_{\lambda_i-i+j}} = 0$ if $\lambda_i - i + j < 0$.

Proof. We obtain this corollary if put $x_i = q^{i-1}$ $(i = 1,2,\cdots)$ in the Jacobi-Trudi identity:

$$s_\lambda = \det\left(h_{\lambda_i-i+j}\right) = \det\left(e_{\lambda'_i-i+j}\right),$$

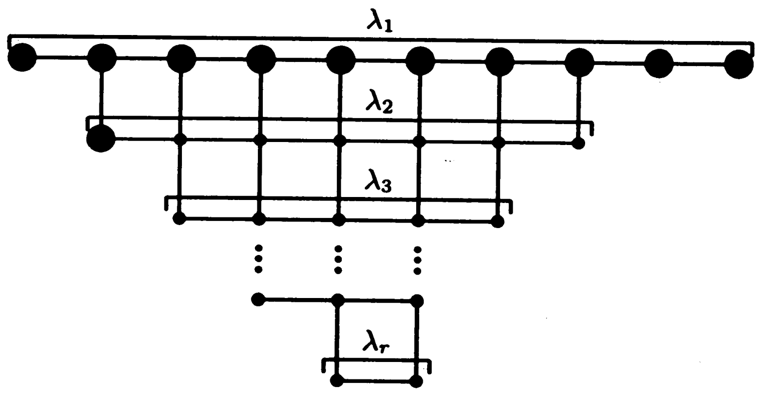
where h_r is the rth complete symmetric function and e_r the rth elementary symmetric function. □

We proved this lemma using the Schur functions to make the proof shorter, but this lemma also can be proven using the Vandermonde determinant without the knowledge of the symmetric functions.

6.2 Shifted Shapes

Although the hook formulas for shifted shapes is well-known, here we briefly review the associated Pfaffian evaluations, which will be used in the following sections.

Shifted Shapes $(\lambda_1 > \lambda_2 > \cdots > \lambda_r \geq 1)$



Lemma 6.3 *Let n be an even intger. Then*

$$\operatorname{pf}\left(\frac{x_i - x_j}{1 - x_i x_j}\right)_{1 \leq i,j \leq n} = \prod_{1 \leq i<j \leq n} \frac{x_i - x_j}{1 - x_i x_j}. \tag{21}$$

Proof This proof of using a residue theorem is suggested by H.Kawamuko. The reader can find another proof in [13]. By the expansion formula of Pfaffian along the first row and column it is enough to show that

$$\sum_{k=1}^{n} \frac{x_k - y}{1 - x_k y} \prod_{\substack{i=1 \\ i \neq k}}^{n} \frac{1 - x_i x_k}{x_i - x_k} = \begin{cases} \prod_{i=1}^{n} \frac{x_i - y}{1 - x_i y} & \text{if } n \text{ is even,} \\ \prod_{i=1}^{n} \frac{x_i - y}{1 - x_i y} - 1 & \text{if } n \text{ is odd.} \end{cases} \tag{22}$$

Let $H(z)$ be the rational function defined by

$$H(z) = \frac{z - y}{1 - yz} \frac{\prod_{i=1}^{n}(1 - x_i z)}{\prod_{i=1}^{n}(x_i - z)} \frac{1}{1 - z^2}.$$

Then each x_k is a simple pole, whose residue is

$$-\frac{x_k - y}{1 - x_k y} \frac{\prod_{i \neq k}(1 - x_i x_k)}{\prod_{i \neq k}(x_i - x_k)}.$$

Nextly $z = y^{-1}$ is also a simple pole with residue

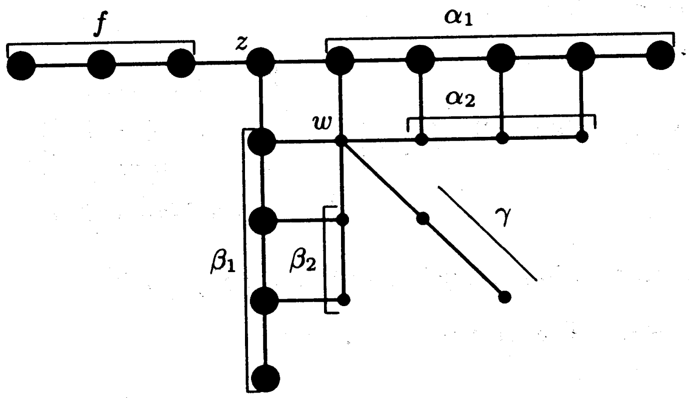
$$\frac{\prod_{i=1}^{n}(x_i - y)}{\prod_{i=1}^{n}(1 - x_i y)}$$

Similarly $+1$, (resp. -1) is a simple pole of residue $(-1)^{n+1}/2$ (resp. $-1/2$). Lastly $H(z)$ is analytic at infinity since $-\lim_{z \to \infty} zH(z) = 0$. Because the sum of the all residues in $\mathbb{C} \cup \{\infty\}$ is 0, we obtain the identity (22). This proves the lemma. □

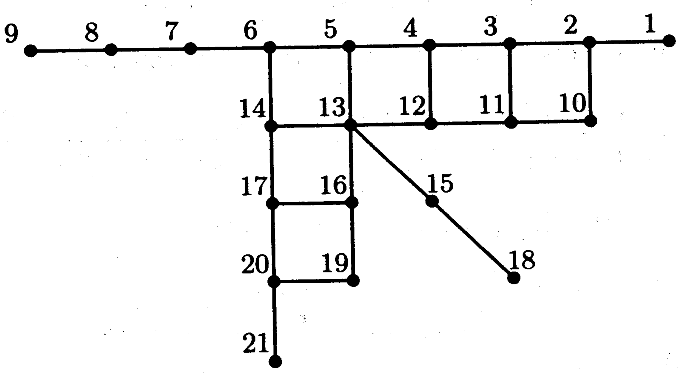
6.3 Birds

The birds case is the simplest. Let $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$ be strict partitions of length 2, and f and γ nonnegative integers which satisfy $\alpha_1 > \alpha_2 \geq 0$, $\beta_1 > \beta_2 \geq 0$ and $f \geq \gamma \geq 0$. By abuse of language we call the poset defined by the following diagram, denoted by $P = P(\alpha, \beta, f, \gamma; 3)$, the birds whereas they are not exactly the same as Proctor defined. The *top tree posets* is the filter which consists of the large solid dots.

Birds ($f \geq \gamma$, $\beta_1 > \beta_2 \geq 0$, $\alpha_1 > \alpha_2 \geq 0$)



Here we fix a labeling of each vertex in which the labels increase from right to left along each row and from top to bottom along each column. An example of such a labeling is given by the following picture.



We call this labeling the "column-strict" labeling. Of course, we can choose other labelings which may give different generating functions, but also serve to prove the hook formulas of d-complete posets. The "column-strict" labeling is one of such a choice and the relations and a generalization of labelings will be studied in [2]. The generating function of (P, ω)-partitions, where ω is the column-strict labeling, is given by

$$\sum q^{z+w}\begin{bmatrix} z+f \\ f \end{bmatrix}_q \frac{1}{\prod_{i=1}^{2}(q;q)_{\alpha_i}} \begin{vmatrix} q^{\alpha_1 z} & q^{\alpha_1 w} \\ q^{\alpha_2 z} & q^{\alpha_2 w} \end{vmatrix} \frac{q^{\sum_{i=1}^{2}\binom{\beta_i+1}{2}}}{\prod_{i=1}^{2}(q;q)_{\beta_i}} \begin{vmatrix} q^{\beta_1 z} & q^{\beta_1 w} \\ q^{\beta_2 z} & q^{\beta_2 w} \end{vmatrix} \frac{q^{\binom{\gamma+1}{2}+\gamma w}}{(q;q)_\gamma} \tag{23}$$

where the sum runs over $0 \leq z \leq w$. The reader who is not skilled with deriving this kind of generating functions should see the next section, where we will expalain the methods in more details. We omit the more explanation about it here because the birds case is an easy and straightforward calculation. For convention we put

$$C=\frac{q^{\;{}^{2}_{i=1}\binom{\beta_i+1}{2}+\binom{\gamma+1}{2}}}{\prod_{i=1}^{2}(q;q)_{\alpha_i}\prod_{i=1}^{2}(q;q)_{\beta_i}(q;q)_{\gamma}}.$$

Then (23) is equal to

$$C\sum_{z=0}^{\infty}\sum_{w=z}^{\infty}q^{z}\begin{bmatrix}z+f\\ f\end{bmatrix}_q\begin{vmatrix}q^{\alpha_1 z} & q^{(\alpha_1+\gamma+1)w} & \begin{vmatrix}q^{\beta_1 z} & q^{\beta_1 w}\\ q^{\beta_2 z} & q^{\beta_2 w}\end{vmatrix}\\ q^{\alpha_2 z} & q^{(\alpha_2+\gamma+1)w} & \begin{vmatrix}q^{\beta_1 z} & q^{\beta_1 w}\\ q^{\beta_2 z} & q^{\beta_2 w}\end{vmatrix}\end{vmatrix}.$$

Taking the summation on w leads to

$$C\sum_{z=0}^{\infty}q^{(|\alpha|+|\beta|+\gamma+2)z}\begin{bmatrix}z+f\\ f\end{bmatrix}_q\begin{vmatrix}1 & \begin{vmatrix}1 & \frac{1}{(\alpha_1+\beta_1+\gamma+1)_q}\\ 1 & \frac{1}{(\alpha_1+\beta_2+\gamma+1)_q}\end{vmatrix}\\ 1 & \begin{vmatrix}1 & \frac{1}{(\alpha_2+\beta_1+\gamma+1)_q}\\ 1 & \frac{1}{(\alpha_2+\beta_2+\gamma+1)_q}\end{vmatrix}\end{vmatrix}.$$

If we take the summation on z using the formula $\sum_{z=0}^{\infty}q^{(|\alpha|+|\beta|+\gamma+2)z}\begin{bmatrix}z+f\\ f\end{bmatrix}_q$ $\frac{(q;q)_{|\alpha|+|\beta|+\gamma+1}}{(q;q)_{|\alpha|+|\beta|+\gamma+f+2}}$, this equation is equal to

$$C\frac{(q;q)_{|\alpha|+|\beta|+\gamma+1}}{(q;q)_{|\alpha|+|\beta|+\gamma+f+2}}q^{\gamma+1}(q^{\beta_2}-q^{\beta_1})\begin{vmatrix}1 & \frac{q^{\alpha_1}}{(\alpha_1+\beta_1+\gamma+1)_q(\alpha_1+\beta_2+\gamma+1)_q}\\ 1 & \frac{q^{\alpha_2}}{(\alpha_2+\beta_1+\gamma+1)_q(\alpha_2+\beta_2+\gamma+1)_q}\end{vmatrix}.$$

Finally if we use the formula

$$q^{y}(x+a)_q(x+b)_q-q^{x}(y+a)_q(y+b)_q=(q^{y}-q^{x})(x+y+a+b)_q,$$

then we obtain the generating function $F(P,\omega;q)$ is equal to

$$C\frac{q^{\gamma+1}(q^{\alpha_2}-q^{\alpha_1})(q^{\beta_2}-q^{\beta_1})(q;q)_{|\alpha|+|\beta|+\gamma+1}(|\alpha|+|\beta|+2\gamma+2)_q}{(q;q)_{|\alpha|+|\beta|+\gamma+f+2}\prod_{i=1}^{2}\prod_{j=1}^{2}(\alpha_i+\beta_j+\gamma+1)_q}.$$

By the equation (3), $\sum_{\pi\in\mathcal{W}(P,\omega)}q^{\mathrm{ch}(\pi_0\circ\pi)}$ is equal to

$$q^{\alpha_2+\binom{\beta_1+1}{2}+\binom{\beta_2+2}{2}+\binom{\gamma+2}{2}-1}\frac{(q;q)_n}{\prod_{i=1}^{2}(q;q)_{\alpha_i}\prod_{i=1}^{2}(q;q)_{\beta_i}(q;q)_{\gamma}}$$
$$\times\frac{(q;q)_{|\alpha|+|\beta|+\gamma+1}(\alpha_1-\alpha_2)_q(\beta_1-\beta_2)_q(|\alpha|+|\beta|+2\gamma+2)_q}{(q;q)_{|\alpha|+|\beta|+\gamma+f+2}\prod_{i=1}^{2}\prod_{j=1}^{2}(\alpha_i+\beta_j+\gamma+1)_q},\quad(24)$$

where $n=\sharp P=|\alpha|+|\beta|+\gamma+f+2$ and π_0 is the longest element in S_n. By a straightforward calculation, it is easy to see that this identity is equal to

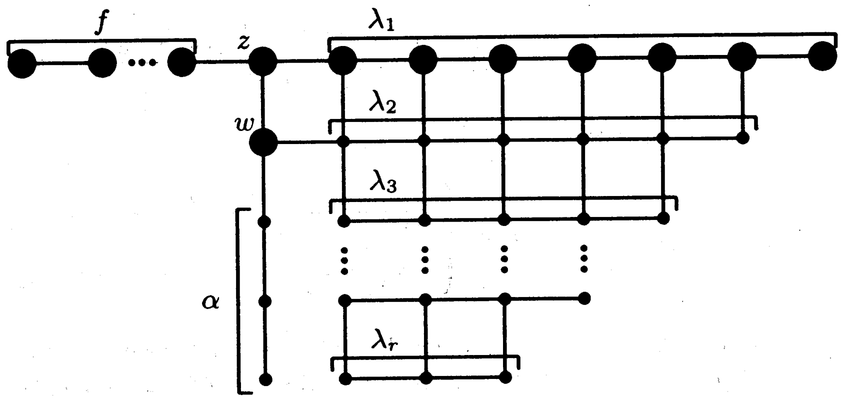
$$q^{n(P,\omega)}\frac{(q;q)_n}{\prod_{x\in P}(q;q)_{h(x)}}.$$

Here we define $n(P,\omega)=\alpha_2+\binom{\beta_1+1}{2}+\binom{\beta_2+2}{2}+\binom{\gamma+2}{2}-1$ for the bird P of the above shape and the above column-strict labeling ω.

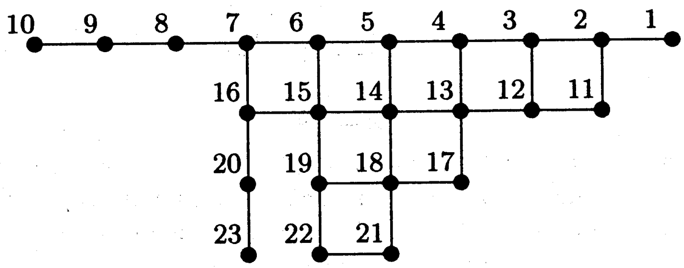
6.4 Insets

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ $(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0)$ be a partition and f and α be positive integers which satisfies $f \geq r-2 \geq 0$ and $\alpha \geq 0$. Then, we call the poset given in the following diagram, denoted by $P = P(\lambda, f, \alpha; 4)$, the Insets. In the diagram elements become bigger if one goes in the north-west direction.

Insets $(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 1,\ f \geq r-2 \geq 0,\ \alpha \geq 0)$



In this subsection we consider two different labelings for P. One is a labeling in which the labels in each vertex increase from right to left along each row and from top to bottom along each column, which is called a *colum-strict* labeling and denoted by ω_c; the other is a labeling in which the labels increase from left to right along each row and from bottom to top along each column, which is called a *row-strict* labeling and denoted by ω_r. An example of a column-strict labeling is given in the following picture.



In this example $f = 3$, $\alpha = 2$ and $\lambda = (6, 5, 3, 2)$.

Theorem 6.4 *Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be a partition, f and α be integers such that $\lambda_1 \geq \cdots \geq \lambda_r \geq 1$, $f \geq r-2 \geq 0$ and $\alpha \geq 0$. Let $P = P(\lambda, f, \lambda; 4)$ be the Insets and let ω_c and ω_r be a column-strict and a row-strict labeling, respectively.*

(i) Then the generating function of (P, ω_c)-partitions is given by

$$F(P, \omega_c; q) = \frac{q^{\binom{\alpha+1}{2}} (q;q)_{|\lambda|+\alpha+1}}{(q;q)_\alpha (q;q)_{|\lambda|+\alpha+f+2}} \det\left(A_{ij}\right)_{1 \leq i,j \leq r},$$

where

$$A_{ij} = \begin{cases} \frac{q^{|\lambda|-\lambda_i+\alpha+i}}{(|\lambda|-\lambda_i+\alpha+i)_q} \frac{1}{(q;q)_{\lambda_i-i+j}} & \text{if } j = 1, \\ \frac{1}{(q;q)_{\lambda_i-i+j}} & \text{if } j = 2, \\ \frac{q^{\lambda_i-i+j}}{(q;q)_{\lambda_i-i+j}} & \text{if } 3 \le j \le r. \end{cases}$$

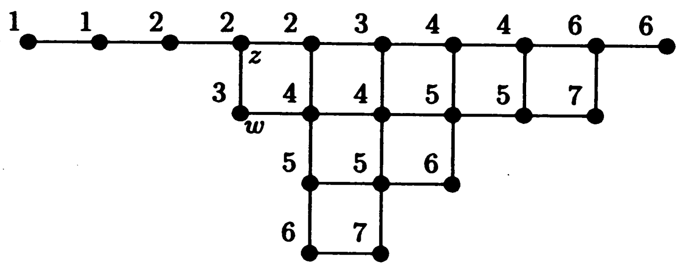
(ii) The generating function of (P, ω_r)-partitions is given by

$$F(P, \omega_r; q) = \frac{q^{\binom{f}{2}+(|\lambda|+\alpha+2)f}(q;q)_{|\lambda|+\alpha+1}}{(q;q)_\alpha (q;q)_{|\lambda|+\alpha+f+2}} \det\left(B_{ij}\right)_{1\le i,j\le r},$$

where

$$B_{ij} = \begin{cases} \frac{1}{(|\lambda|-\lambda_i+\alpha+i)_q} \frac{q^{\binom{\lambda_i-i+j+1}{2}}}{(q;q)_{\lambda_i-i+j}} & \text{if } j = 1, \\ \frac{q^{\binom{\lambda_i-i+j+1}{2}}}{(q;q)_{\lambda_i-i+j}} & \text{if } 2 \le j \le r. \end{cases}$$

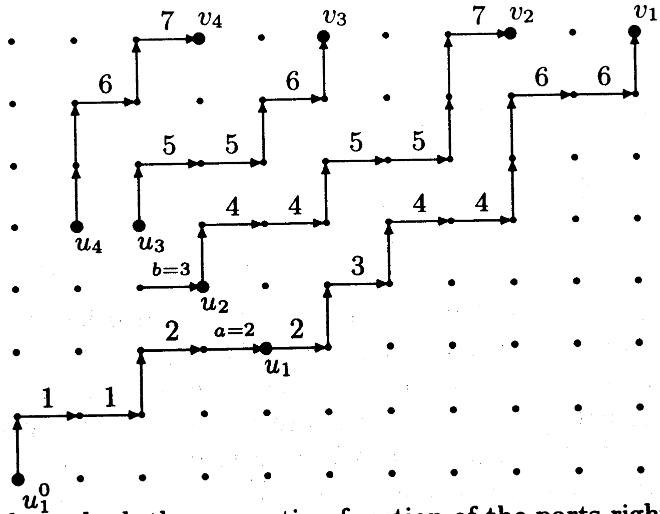
Proof. First we consider the column-strict case. As proved in [10], this poset $P = P(\lambda, f, \alpha; 4)$ contains the top tree $T = Y(f; g, h)$ with $g = \lambda_1$ and $h = 1$. Let z be the element which locates in the branching point of this top tree, i.e. z is the element designated in the above picture. Let w be the element just below z which is designated in the above picture. Let a and b be integers such that $0 \le a < b$, let $\mathcal{A}_{\omega_c}(\lambda, f, \alpha; a, b)$ denote the set of all the (P, ω_c)-partitions σ satisfying $\sigma(z) = a$ and $\sigma(w) = b$. An example of column-strict (P, ω_c)-partition with $a = 2$ and $b = 3$ is shown below.



To obtain the generating function of $\mathcal{A}_{\omega_c}(\lambda, f, \alpha; a, b)$, we translate the (P, ω_c)-partition σ into a lattice path. Define a directed graph D_c on the vertex set $\mathbb{N}^2$ with an edge directed from u to v whenever $v - u = (1, 0)$ or $(0, 1)$. For $u = (i, j)$, we assign the weight q^j to the edge $u \to v$ if $v - u = (1, 0)$ and the weight 1 if $v - u = (0, 1)$. Choose an integer $N \ge 0$, and let $\boldsymbol{v} = (v_1, \dots, v_r)$ be the r-vertex, in which $v_i = (\lambda_i - i, N)$ for $i = 1, \dots, r$. Let $\boldsymbol{u} = (u_1, \dots, u_r)$ be the r-vertex, in which each u_j is given by

$$u_j = \begin{cases} (-j, a) & \text{if } j = 1, \\ (-j, b) & \text{if } j = 2, \\ (-j, b+1) & \text{if } j = 3, \dots, r. \end{cases}$$

Let $u_1^0 = (-2 - f, 0)$. Then, for a (P, ω_c)-partition σ in $\mathcal{A}_{\omega_c}(\lambda, f, \alpha; a, b)$, the parts of σ which are right to z and w can be interpreted as an r-path which connect r-vertex $\boldsymbol{u}$ to r-vertex $\boldsymbol{v}$. For example, the above (P, ω_c)-partition σ corresponds to the following lattice path.



By the lattice path method, the generating function of the parts right to z and w of $\sigma \in \mathcal{A}_{\omega_c}(\lambda, f, \alpha; a, b)$, which correspond to the r-paths that connect r-vertices $\boldsymbol{u}$ and $\boldsymbol{v}$, is given by $\det A'$, where $A' = (A'_{ij})_{1\le i,j\le r}$ is defined by

$$A'_{ij} = \begin{cases} q^{(\lambda_i-i+j)a}\begin{bmatrix} \lambda_i-i+j+N-a \\ \lambda_i-i+j \end{bmatrix}_q & \text{if } j=1, \\ q^{(\lambda_i-i+j)b}\begin{bmatrix} \lambda_i-i+j+N-b \\ \lambda_i-i+j \end{bmatrix}_q & \text{if } j=2, \\ q^{(\lambda_i-i+j)(b+1)}\begin{bmatrix} \lambda_i-i+j+N-b-1 \\ \lambda_i-i+j \end{bmatrix}_q & \text{if } j=3,\dots,r. \end{cases}$$

So the generating function of (P, ω_c)-partitions is given by

$$F(P, \omega_c; q) = \sum \begin{bmatrix} a+f \\ f \end{bmatrix}_q q^{a+b} \frac{q^{\binom{\alpha+1}{2}+\alpha b}}{(q;q)_\alpha} \det A'',$$

where the sum runs over all integers a, b such that $0 \le a < b$ and $A''_{ij} = \lim_{N\to\infty} A'_{ij}$ is given by

$$A''_{ij} = \begin{cases} \dfrac{q^{(\lambda_i-i+j)a}}{(q;q)_{\lambda_i-i+j}} & \text{if } j=1, \\ \dfrac{q^{(\lambda_i-i+j)b}}{(q;q)_{\lambda_i-i+j}} & \text{if } j=2, \\ \dfrac{q^{(\lambda_i-i+j)(b+1)}}{(q;q)_{\lambda_i-i+j}} & \text{if } j=3,\dots,r. \end{cases}$$

Now we expand $\det A''$ along the first column, then we obtain $q^{a+(\alpha+1)b} \det A'' = \det A'''$ with $A''' = (A'''_{ij})_{1\le i,j\le r}$ given by

$$A'''_{ij} = \begin{cases} \dfrac{q^{(\lambda_i-i+j+1)a+(|\lambda|-\lambda_i+i+\alpha)b}}{(q;q)_{\lambda_i-i+j}} & \text{if } j=1, \\ \dfrac{1}{(q;q)_{\lambda_i-i+j}} & \text{if } j=2, \\ \dfrac{q^{\lambda_i-i+j}}{(q;q)_{\lambda_i-i+j}} & \text{if } j=3,\dots,r. \end{cases}$$

Thus we take the sum on b from a to infinity, then the above sum is equal to

$$F(P, \omega_c; q) = \frac{q^{\binom{\alpha+1}{2}}}{(q;q)_\alpha} \sum_{a=0}^{\infty} \begin{bmatrix} a+f \\ f \end{bmatrix}_q q^{(|\lambda|+\alpha+2)a} \det(A_{ij})_{1\le i,j\le r}.$$

If we take the sum on a using the formula $\sum_{a=0}^{\infty} \begin{bmatrix} a+f \\ f \end{bmatrix}_q q^{(|\lambda|+\alpha+2)a} = \frac{(q;q)_{|\lambda|+\alpha+1}}{(q;q)_{|\lambda|+\alpha+f+2}}$, we obtain (i).

To prove (ii), we take the elements z and w in P as before. Let $\sigma \in \mathcal{A}_{\omega_r}(\lambda, f, \alpha; a, b)$ denote the set of (P, ω_r)-partitions σ which satisfy $\sigma(z) = a$ and $\sigma(w) = b$ for $0 \le a \le b$. This time, we define a directed graph D_r on the vertex set $\mathbb{N}^2$ with an edge directed from u to v whenever $v - u = (1,1)$ or $(0,1)$. As before, for $u = (i,j)$, we assign the weight q^j to the edge $u \to v$ if $v - u = (1,1)$ and the weight 1 if $v - u = (0,1)$.

Choose an integer $N \geq 0$, and let $\boldsymbol{v} = (v_1, \ldots, v_r)$ be the r-vertex defined $v_i = (\lambda_i - i, N+1)$ for $i = 1, \ldots, r$. Let $\boldsymbol{u} = (u_1, \ldots, u_r)$ be the r-vertex defined by

$$u_j = \begin{cases} (-j, a+1) & \text{if } j = 1, \\ (-j, b+1) & \text{if } j = 2, \ldots, r. \end{cases}$$

Let $u_1^0 = (-2-f, 0)$. Then each (P, ω_r)-partition $\sigma \in \mathcal{A}_{\omega_r}(\lambda, f, \alpha; a, b)$ correspond bijectively to an r-path $P = (p_1, \cdots, p_r)$ such that p_1 connects u_1^0 to v_1 via u_1, and each P_i connects u_i to v_i for $i = 2, \cdots, r$. By the lattice path method, the generating function of the parts right to z and w of $\sigma \in \mathcal{A}_{\omega_r}(\lambda, f, \alpha; a, b)$, which correspond to the r-paths that connect r-vertices $\boldsymbol{u}$ and $\boldsymbol{v}$, is given by $\det B'$, where $B' = (B'_{ij})_{1 \leq i,j \leq r}$ is given by

$$B'_{ij} = \begin{cases} q^{(\lambda_i - i + j)a + \binom{\lambda_i - i + j + 1}{2}} \begin{bmatrix} N-a \\ \lambda_i - i + j \end{bmatrix}_q & \text{if } j = 1, \\ q^{(\lambda_i - i + j)b + \binom{\lambda_i - i + j + 1}{2}} \begin{bmatrix} N-b \\ \lambda_i - i + j \end{bmatrix}_q & \text{if } j = 2, \ldots, r. \end{cases}$$

So the generating function of (P, ω_c)-partitions is given by

$$F(P, \omega_r; q) = \sum q^{\binom{f}{2}} \begin{bmatrix} a \\ f \end{bmatrix}_q q^{a+b} \frac{q^{ab}}{(q;q)_a} \det B'',$$

where the sum runs over all integers a, b such that $0 \leq a \leq b$ and $B''_{ij} = \lim_{N \to \infty} B'_{ij}$ is given by

$$B''_{ij} = \begin{cases} \frac{q^{(\lambda_i - i + j)a + \binom{\lambda_i - i + j + 1}{2}}}{(q;q)_{\lambda_i - i + j}} & \text{if } j = 1, \\ \frac{q^{(\lambda_i - i + j)b + \binom{\lambda_i - i + j + 1}{2}}}{(q;q)_{\lambda_i - i + j}} & \text{if } j = 2, \ldots, r. \end{cases}$$

Now we expand $\det B''$ along the first column, and an easy calculation leads to $\det B'' = \det B'''$, where $B''' = (B'''_{ij})_{1 \leq i,j \leq r}$ is given by

$$B'''_{ij} = \begin{cases} \frac{q^{(\lambda_i - i + j)a + (|\lambda| - \lambda_i + i - 1)b + \binom{\lambda_i - i + j + 1}{2}}}{(q;q)_{\lambda_i - i + j}} & \text{if } j = 1, \\ \frac{q^{\binom{\lambda_i - i + j + 1}{2}}}{(q;q)_{\lambda_i - i + j}} & \text{if } j = 2, \ldots, r. \end{cases}$$

Thus we have

$$F(P, \omega_r; q) = \frac{q^{\binom{f}{2}}}{(q;q)_a} \sum_{0 \leq a \leq b} \begin{bmatrix} a \\ f \end{bmatrix}_q \det B'''',$$

where

$$B''''_{ij} = \begin{cases} \frac{q^{(\lambda_i - i + j + 1)a + (|\lambda| - \lambda_i + i + \alpha)b + \binom{\lambda_i - i + j + 1}{2}}}{(q;q)_{\lambda_i - i + j}} & \text{if } j = 1, \\ \frac{q^{\binom{\lambda_i - i + j + 1}{2}}}{(q;q)_{\lambda_i - i + j}} & \text{if } j = 2, \ldots, r. \end{cases}$$

Now we take the sum on b first, then take the on a using $\sum_{a \geq 0} \begin{bmatrix} a \\ f \end{bmatrix}_q q^{(|\lambda| + \alpha + 2)(}$ $\frac{(q;q)_{|\lambda| + \alpha + 1}}{(q;q)_{|\lambda| + \alpha + f + 2}}$. This proves (ii). $\square$

Lemma 6.5 *Let r, s be integers such that $r \geq s \geq 0$, and let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be a partition. Let $C^{(s)} = \left(C_{ij}^{(s)}\right)_{1 \leq i,j \leq r}$ be the r by r matrix whose entries are given by*

$$C_{ij}^{(s)} = \begin{cases} \frac{1}{(q;q)_{\lambda_i - i + j}} & \text{if } 1 \leq j \leq s, \\ \frac{q^{\lambda_i - i + j}}{(q;q)_{\lambda_i - i + j}} & \text{if } s + 1 \leq j \leq r. \end{cases}$$

Here we use the convention $\frac{1}{(q;q)_{\lambda_i - i + j}} = 0$ if $\lambda_i - i + j < 0$. Then

$$\det C^{(s)} = \begin{cases} q^{n(\lambda)+|\lambda|} \frac{\prod_{1 \leq i < j \leq r} (\lambda_i - \lambda_j - i + j)_q}{\prod_{i=1}^{r} (q;q)_{\lambda_i + r - i}} & \text{if } s = 0, \\ q^{n(\lambda)} \frac{\prod_{1 \leq i < j \leq r} (\lambda_i - \lambda_j - i + j)_q}{\prod_{i=1}^{r} (q;q)_{\lambda_i + r - i}} & \text{if } 1 \leq s \leq r. \end{cases}$$

Proof. We proceed with reverse induction on s. If $s = r$, then, we already obtained this identity in Corollary 6.2. Assume this identity holds for $s \geq k \geq 2$. If we subtract each (i,k)-entry from each $(i, k-1)$-entry of $C^{(k)}$ for $i = 1, \cdots, r$, the determinant remains invariant. Thus we obtain the $s = k - 1$ case because of

$$\frac{1}{(q;q)_{\lambda_i - i + j}} - \frac{1}{(q;q)_{\lambda_i - i + j - 1}} = \frac{q^{\lambda_i - i + j}}{(q;q)_{\lambda_i - i + j}}.$$

This proves the identity for $1 \leq s \leq r$. The $s = 0$ case immediately follows from the $s = r$ case since $\det(q^{\lambda_i - i + j} a_{ij}) = q^{\sum \lambda_i} \det(a_{ij})$ for any matrix (a_{ij}). □

Lemma 6.6 *Let $r \geq 1$ be a positive integer and s be an integer such that $1 \leq s \leq r$. Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be a partition, and α be a nonegative integer.*

(i) Let $A^{(s)} = \left(A_{ij}^{(s)}\right)_{1 \leq i,j \leq r}$ be the r by r matrix defined by

$$A_{ij}^{(s)} = \begin{cases} \frac{q^{|\lambda| - \lambda_i + i - 1}}{(|\lambda| - \lambda_i + i + \alpha)_q} \frac{1}{(q;q)_{\lambda_i - i + j}} & \text{if } j = 1, \\ \frac{1}{(q;q)_{\lambda_i - i + j}} & \text{if } 2 \leq j \leq s, \\ \frac{q^{\lambda_i - i + j}}{(q;q)_{\lambda_i - i + j}} & \text{if } s + 1 \leq j \leq r. \end{cases}$$

Then we have

$$\det A^{(s)}$$

$$= \begin{cases} q^{n(\lambda) - \alpha - 1} \frac{\prod_{i=2}^{r} (|\lambda| + \alpha + i)_q - \prod_{i=1}^{r} (|\lambda| - \lambda_i + \alpha + i)_q}{\prod_{i=1}^{r} (|\lambda| - \lambda_i + i + \alpha)_q} \frac{\prod_{1 \leq i < j \leq r} (\lambda_i - \lambda_j - i + j)_q}{\prod_{i=1}^{r} (q;q)_{\lambda_i + r - i}} & \text{if } s = 1, \\ q^{n(\lambda)} \frac{\prod_{i=2}^{r} (|\lambda| + \alpha + i)_q}{\prod_{i=1}^{r} (|\lambda| - \lambda_i + i + \alpha)_q} \frac{\prod_{1 \leq i < j \leq r} (\lambda_i - \lambda_j - i + j)_q}{\prod_{i=1}^{r} (q;q)_{\lambda_i + r - i}} & \text{if } s \geq 2. \end{cases}$$

(ii) Let $B = \left(B_{ij}\right)_{1 \leq i,j \leq r}$ be the $r \times r$ matrix defined by

$$B_{ij} = \begin{cases} \frac{1}{(|\lambda| - \lambda_i + \alpha + i)_q} \frac{q^{\binom{\lambda_i - i + j + 1}{2}}}{(q;q)_{\lambda_i - i + j}} & \text{if } j = 1, \\ \frac{q^{\binom{\lambda_i - i + j + 1}{2}}}{(q;q)_{\lambda_i - i + j}} & \text{if } 2 \leq j \leq r. \end{cases}$$

Then we have

$$\det B$$

$$= q^{\sum_{i=1}^{r} \binom{\lambda_i + 1}{2}} \frac{\prod_{i=2}^{r} (|\lambda| + \alpha + i)_q}{\prod_{i=1}^{r} (|\lambda| - \lambda_i + \alpha + i)_q} \frac{\prod_{1 \leq i < j \leq r} (\lambda_i - \lambda_j - i + j)_q}{\prod_{i=1}^{r} (q;q)_{\lambda_i + r - i}}.$$

Here we also use the convention $\frac{1}{(q;q)_{\lambda_i-i+j}}=0$ *if* $\lambda_i-i+j<0$.

Proof. We first consider (i). By expanding $\det A^{(s)}$ along the first column we obtain

$$\det A^{(s)}=\sum_{k=1}^{r}(-1)^{k+1}\frac{q^{|\lambda|-\lambda_k+k-1}}{(|\lambda|-\lambda_k+k+\alpha)_q}\frac{1}{(q;q)_{\lambda_k-k+1}}\det A^{(s),k,1}.$$

By the above lemma we have

$$\det A^{(s),k,1}$$

$$=\begin{cases} q^{n(\lambda^{(k)})+|\lambda^{(k)}|}\frac{\prod_{1\le i<j\le r-1}(\lambda_i^{(k)}-\lambda_j^{(k)}-i+j)_q}{\prod_{i=1}^{r-1}(q;q)_{\lambda_i^{(k)}+r-1-i}} & \text{if } s=1,\\ q^{n(\lambda^{(k)})}\frac{\prod_{1\le i<j\le r-1}(\lambda_i^{(k)}-\lambda_j^{(k)}-i+j)_q}{\prod_{i=1}^{r-1}(q;q)_{\lambda_i^{(k)}+r-1-i}} & \text{if } s\ge 2,\end{cases}$$

where

$$\lambda_i^{(k)}=\begin{cases}\lambda_i+1 & \text{if } 1\le i\le k-1,\\ \lambda_{i+1} & \text{if } k\le i\le r-1.\end{cases}$$

By substituting $\lambda^{(k)}$, a direct calculation leads to

$$q^{-n(\lambda)}\frac{\prod_{i=1}^{r}(q;q)_{\lambda_i-i+r}}{\prod_{1\le i<j\le r}(\lambda_i-\lambda_j-i+j)_q}\det A^{(s)}$$

$$=\begin{cases}\sum_{k=1}^{r}\frac{q^{|\lambda|-\lambda_k+k-1}}{(|\lambda|-\lambda_k+k+\alpha)_q}\frac{\prod_{i=2}^{r}(\lambda_k-k+i)_q}{\prod_{\substack{i=1\\ i\ne k}}^{r}(\lambda_k-\lambda_i-k+i)_q} & \text{if } s=1,\\ \sum_{k=1}^{r}\frac{1}{(|\lambda|-\lambda_k+k+\alpha)_q}\frac{\prod_{i=2}^{r}(\lambda_k-k+i)_q}{\prod_{\substack{i=1\\ i\ne k}}^{r}(\lambda_k-\lambda_i-k+i)_q} & \text{if } s\ge 2.\end{cases}$$

Thus, to prove our theorem, we need to show the following identities.

$$\sum_{k=1}^{r}q^{|\lambda|-\lambda_k+k-1}\prod_{\substack{i=1\\ i\ne k}}^{r}(|\lambda|-\lambda_i+i+\alpha)_q\frac{\prod_{i=2}^{r}(\lambda_k-k+i)_q}{\prod_{\substack{i=1\\ i\ne k}}^{r}(\lambda_k-\lambda_i-k+i)_q}$$

$$=q^{-\alpha-1}\left\{\prod_{i=2}^{r}(|\lambda|+\alpha+i)_q-\prod_{i=1}^{r}(|\lambda|-\lambda_i+\alpha+i)_q\right\}.$$

$$\sum_{k=1}^{r}\prod_{\substack{i=1\\ i\ne k}}^{r}(|\lambda|-\lambda_i+i+\alpha)_q\frac{\prod_{i=2}^{r}(\lambda_k-k+i)_q}{\prod_{\substack{i=1\\ i\ne k}}^{r}(\lambda_k-\lambda_i-k+i)_q}=\prod_{i=2}^{r}(|\lambda|+\alpha+i)_q. \tag{25}$$

If we regard each side of these identities as a polynomial of q^{α}, then the both sides are polynomials of degree $r-1$. So, to see the both sides coinside, it is enough to see it on r distinct values of α. Thus, if we substitute $\alpha=-|\lambda|+\lambda_k-k$ for $k=1,\cdots,r$, it is immediate to see the both sides are equal to each other. This proves (i).

Next we prove (ii) in a similar method as (i). If we expand $\det B$ along the first column, then we have

$$\det B=\sum_{k=1}^{r}\frac{(-1)^{k+1}}{(|\lambda|-\lambda_k+\alpha+k)_q}\frac{q^{\binom{\lambda_k-k+2}{2}}}{(q;q)_{\lambda_k-k+1}}\det B^{k,1}.$$

By Corollary 6.2, we have

$$\det B^{k,1} = q^{\sum_{i=1}^{r-1}\binom{\lambda_i^{(k)}+1}{2}} \frac{\prod_{1\le i<j\le r-1}(\lambda_i^{(k)}-\lambda_j^{(k)}-i+j)_q}{\prod_{i=1}^{r-1}(q;q)_{\lambda_i^{(k)}+r-1-i}}.$$

Here $\lambda^{(k)}$ is as above. Now, if we substitute $\lambda^{(k)}$, then a straightforward calculation leads to

$$\frac{\prod_{i=1}^{r}(q;q)_{\lambda_i-i+r}}{\prod_{1\le i<j\le r}(\lambda_i-\lambda_j-i+j)_q}\det B$$
$$=\sum_{k=1}^{r}\frac{q^{\sum_{i=1}^{k-1}\binom{\lambda_i+2}{2}+\binom{\lambda_k-k+2}{2}+\sum_{i=k+1}^{r}\binom{\lambda_i+1}{2}+\sum_{i=1}^{k-1}(\lambda_k-\lambda_i-k+i)}}{(|\lambda|-\lambda_k+k+\alpha)_q}\frac{\prod_{i=2}^{r}(\lambda_k-k+i)_q}{\prod_{\substack{i=1\\ i\ne k}}^{r}(\lambda_k-\lambda_i-k+i)_q}.$$

Thus, to prove (ii), it is enough to show that

$$\sum_{k=1}^{r}\frac{q^{\sum_{i=1}^{k-1}\binom{\lambda_i+2}{2}+\binom{\lambda_k-k+2}{2}+\sum_{i=k+1}^{r}\binom{\lambda_i+1}{2}+\sum_{i=1}^{k-1}(\lambda_k-\lambda_i-k+i)}}{(|\lambda|-\lambda_k+k+\alpha)_q}\frac{\prod_{i=2}^{r}(\lambda_k-k+i)_q}{\prod_{\substack{i=1\\ i\ne k}}^{r}(\lambda_k-\lambda_i-k+i)_q}$$
$$=q^{\sum_{i=1}^{r}\binom{\lambda_i+1}{2}}\frac{\prod_{i=2}^{r}(|\lambda|+i+\alpha)_q}{\prod_{i=1}^{r}(|\lambda|-\lambda_i+i+\alpha)_q}$$

If we use $\binom{\lambda_k+1}{2}=\binom{\lambda_k-k+2}{2}+(k-1)\lambda_k-\binom{k-1}{2}$, then this identity is precisely the same one as (25), which is already proven in (i). This proves our lemma. □ The following corollary is an immediate consequence of Theorem 6.4 and Lemma 6.6.

Corollary 6.7 *Let $P = P(\lambda, f, \alpha; 4)$ denote Insets. Let ω_c (resp. ω_r) be a column-strict (resp. row-strict) labeling of P. Then we have*

$$\begin{aligned} f(P,\omega_c;q) =& q^{\binom{\alpha+2}{2}+n(\lambda)}\frac{(q;q)_{|\lambda|+\alpha+1}}{(q;q)_\alpha(q;q)_{|\lambda|+\alpha+f+2}} \\ &\times\frac{\prod_{i=2}^{r}(|\lambda|+\alpha+i)_q}{\prod_{i=1}^{r}(|\lambda|-\lambda_i+i+\alpha)_q}\frac{\prod_{1\le i<j\le r}(\lambda_i-\lambda_j-i+j)_q}{\prod_{i=1}^{r}(q;q)_{\lambda_i+r-i}} \\ f(P,\omega_r;q) =& q^{\binom{f}{2}+f(\alpha+2)+(f+1)|\lambda|+\sum_{i=1}^{r}\binom{\lambda_i}{2}}\frac{(q;q)_{|\lambda|+\alpha+1}}{(q;q)_\alpha(q;q)_{|\lambda|+\alpha+f+2}}, \\ &\times\frac{\prod_{i=2}^{r}(|\lambda|+\alpha+i)_q}{\prod_{i=1}^{r}(|\lambda|-\lambda_i+i+\alpha)_q}\frac{\prod_{1\le i<j\le r}(\lambda_i-\lambda_j-i+j)_q}{\prod_{i=1}^{r}(q;q)_{\lambda_i+r-i}}. \end{aligned}$$

This corollary shows that $\sum_{\pi\in\mathcal{W}(P,\omega)}q^{\mathrm{ch}(\pi_0\circ\pi)}$ is equal to

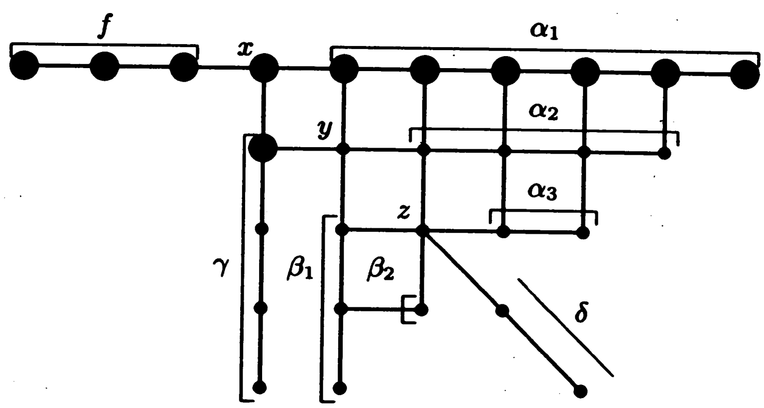
$$q^{n(P,\omega)}\frac{(q;q)_n}{\prod_{x\in P}(q;q)_{h(x)}}$$

for $\omega=\omega_c,\omega_r$ by an easy calculation. Here $n=\sharp P=|\lambda|+\alpha+f+2$, $n(P,\omega_c)=\binom{\alpha+2}{2}+n(\lambda)$ and $n(P,\omega_r)=\binom{f}{2}+(\alpha+2)f+(f+1)|\lambda|+\sum_{i=1}^{r}\binom{\lambda_i}{2}$.

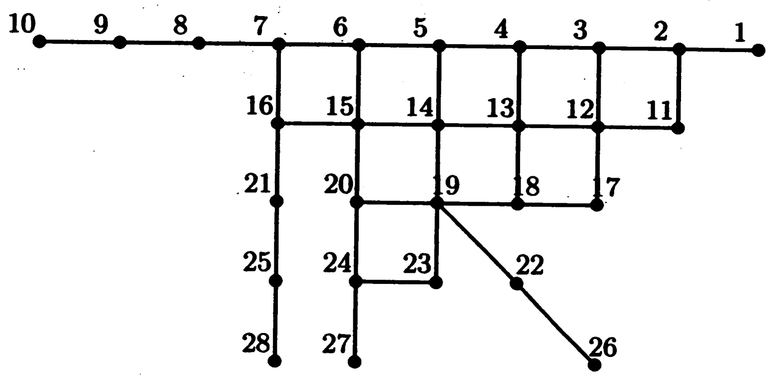
6.5 Tailed Insets

Next we consider Tailed Insets case. Let $\alpha=(\alpha_1,\alpha_2,\alpha_3)$ and $\beta=(\beta_1,\beta_2)$ be strict partitions of length 3 and 2, respectively, i.e. $\alpha_1>\alpha_2>\alpha_3\ge 0$ and $\beta_1>\beta_2\ge 0$. Let $f\ge 2$, γ and δ be nonnegative integers which satisfy $f\ge\gamma\ge 1$. In this paper we call the poset defined by the following diagram, denoted by $P = P(\alpha,\beta,f,\gamma,\delta;5)$, the Tailed Insets. The *top tree posets* is the filter which consists of the large solid dots as before.

Tailed Insets $(f \geq \gamma \geq 1,\ \beta_1 > \beta_2 \geq 0,\ \alpha_1 > \alpha_2 > \alpha_3 \geq 0)$



Here we define a column-strict labeling. The following picture gives an example of a column-strict labeling ω_c.



Theorem 6.8 *Let α, β, γ, δ and f be as above. Let P be the poset given by the above diagram and ω_c a labeling as above. If $\gamma = \delta$, then the generating function $F(q)$ of $(P;\omega_c)$-partitions is written as*

$$\frac{q^{\sum_j \binom{\beta_j+1}{2}+\binom{\gamma+1}{2}+\binom{\gamma+3}{2}+|\beta|}}{\prod_i (q;q)_{\alpha_i}\prod_j (q;q)_{\beta_j}(q;q)_\gamma (q;q)_{\gamma-1}} \cdot \frac{(q;q)_{|\alpha|+|\beta|+\gamma+\delta+2}}{(q;q)_{|\alpha|+|\beta|+\gamma+\delta+f+3}}$$
$$\times \frac{\prod_{1\leq i<j\leq 3}(q^{\alpha_j}-q^{\alpha_i})(q^{\beta_2}-q^{\beta_1})\prod_{i=1}^{2}(\beta_i+|\alpha|+|\beta|+2\gamma+3)_q}{\prod_{i=1}^{3}(|\alpha|-\alpha_i+|\beta|+\gamma+2)_q\prod_{i=1}^{3}\prod_{j=1}^{2}(\alpha_i+\beta_j+\gamma+1)_q}.$$

Proof. From Lemma 4.1, we have

$$\frac{q^{\sum_j \binom{\beta_j+1}{2}+\binom{\gamma+1}{2}+\binom{\delta+1}{2}}}{\prod_i (q;q)_{\alpha_i}\prod_j (q;q)_{\beta_j}(q;q)_\gamma (q;q)_\delta}\sum_{0\leq x<y<z}\begin{bmatrix}x+f\\ f\end{bmatrix} q^{x+y+z}\begin{vmatrix}q^{\gamma x} & q^{\gamma y}\\ 1 & 1\end{vmatrix}$$
$$\times \begin{vmatrix}q^{\alpha_1 x} & q^{\alpha_1 y} & q^{\alpha_1 z}\\ q^{\alpha_2 x} & q^{\alpha_2 y} & q^{\alpha_2 z}\\ q^{\alpha_3 x} & q^{\alpha_3 y} & q^{\alpha_3 z}\end{vmatrix}\begin{vmatrix}q^{\beta_1 y} & q^{\beta_1 z}\\ q^{\beta_2 y} & q^{\beta_2 z}\end{vmatrix} q^{\delta z}$$

This equals

$$C_0 \sum_{0\le x\le y\le z} \begin{bmatrix} x+f \\ f \end{bmatrix} q^{x+y} \begin{vmatrix} q^{\gamma x} & q^{\gamma y} \\ 1 & 1 \end{vmatrix} \begin{vmatrix} q^{\alpha_1 x} & q^{\alpha_1 y} & \begin{vmatrix} q^{\beta_1 y} & q^{(\alpha_1+\beta_1+\delta+1)z} \\ q^{\beta_2 y} & q^{(\alpha_1+\beta_2+\delta+1)z} \end{vmatrix} \\ q^{\alpha_2 x} & q^{\alpha_2 y} & \begin{vmatrix} q^{\beta_1 y} & q^{(\alpha_2+\beta_1+\delta+1)z} \\ q^{\beta_2 y} & q^{(\alpha_2+\beta_2+\delta+1)z} \end{vmatrix} \\ q^{\alpha_3 x} & q^{\alpha_3 y} & \begin{vmatrix} q^{\beta_1 y} & q^{(\alpha_3+\beta_1+\delta+1)z} \\ q^{\beta_2 y} & q^{(\alpha_3+\beta_2+\delta+1)z} \end{vmatrix} \end{vmatrix},$$

where

$$C_0 = \frac{q^{\sum_{j=1}^{2}\binom{\beta_j+1}{2}+\binom{\gamma+1}{2}+\binom{\delta+1}{2}}}{\prod_{i=1}^{3}(q;q)_{\alpha_i}\prod_{j=1}^{2}(q;q)_{\beta_j}(q;q)_\gamma(q;q)_\delta}.$$

By taking the sum on z, we obtain the resulting formula

$$C_0 \sum_{0\le x\le y} \begin{bmatrix} x+f \\ f \end{bmatrix} q^{x+y} \begin{vmatrix} q^{\gamma x} & q^{\gamma y} \\ 1 & 1 \end{vmatrix} \begin{vmatrix} q^{\alpha_1 x} & q^{\alpha_1 y} & \begin{vmatrix} q^{\beta_1 y} & \frac{q^{(\alpha_1+\beta_1+\delta+1)y}}{(\alpha_1+\beta_1+\delta+1)_q} \\ q^{\beta_2 y} & \frac{q^{(\alpha_1+\beta_2+\delta+1)y}}{(\alpha_1+\beta_2+\delta+1)_q} \end{vmatrix} \\ q^{\alpha_2 x} & q^{\alpha_2 y} & \begin{vmatrix} q^{\beta_1 y} & \frac{q^{(\alpha_2+\beta_1+\delta+1)y}}{(\alpha_2+\beta_1+\delta+1)_q} \\ q^{\beta_2 y} & \frac{q^{(\alpha_2+\beta_2+\delta+1)y}}{(\alpha_2+\beta_2+\delta+1)_q} \end{vmatrix} \\ q^{\alpha_3 x} & q^{\alpha_3 y} & \begin{vmatrix} q^{\beta_1 y} & \frac{q^{(\alpha_3+\beta_1+\delta+1)y}}{(\alpha_3+\beta_1+\delta+1)_q} \\ q^{\beta_2 y} & \frac{q^{(\alpha_3+\beta_2+\delta+1)y}}{(\alpha_3+\beta_2+\delta+1)_q} \end{vmatrix} \end{vmatrix},$$

and this is equal to

$$C_0 \sum_{0\le x\le y} \begin{bmatrix} x+f \\ f \end{bmatrix} q^{x+y} \begin{vmatrix} q^{\gamma x} & q^{\gamma y} \\ 1 & 1 \end{vmatrix} \begin{vmatrix} q^{\alpha_1 x+(\alpha_2+\alpha_3+|\beta|+\delta+1)y} & 1 & \begin{vmatrix} 1 & \frac{1}{(\alpha_1+\beta_1+\delta+1)_q} \\ 1 & \frac{1}{(\alpha_1+\beta_2+\delta+1)_q} \end{vmatrix} \\ q^{\alpha_2 x+(\alpha_1+\alpha_3+|\beta|+\delta+1)y} & 1 & \begin{vmatrix} 1 & \frac{1}{(\alpha_2+\beta_1+\delta+1)_q} \\ 1 & \frac{1}{(\alpha_2+\beta_2+\delta+1)_q} \end{vmatrix} \\ q^{\alpha_3 x+(\alpha_1+\alpha_2+|\beta|+\delta+1)y} & 1 & \begin{vmatrix} 1 & \frac{1}{(\alpha_3+\beta_1+\delta+1)_q} \\ 1 & \frac{1}{(\alpha_3+\beta_2+\delta+1)_q} \end{vmatrix} \end{vmatrix}.$$

This is equal to

$$C_0 \sum_{0\le x\le y} \begin{bmatrix} x+f \\ f \end{bmatrix} q^{x} \begin{vmatrix} q^{\alpha_1 x} \begin{vmatrix} q^{\gamma x} & q^{(\alpha_2+\alpha_3+|\beta|+\gamma+\delta+2)y} \\ 1 & q^{(\alpha_2+\alpha_3+|\beta|+\delta+2)y} \end{vmatrix} & 1 & \begin{vmatrix} 1 & \frac{1}{(\alpha_1+\beta_1+\delta+1)_q} \\ 1 & \frac{1}{(\alpha_1+\beta_2+\delta+1)_q} \end{vmatrix} \\ q^{\alpha_2 x} \begin{vmatrix} q^{\gamma x} & q^{(\alpha_1+\alpha_3+|\beta|+\gamma+\delta+2)y} \\ 1 & q^{(\alpha_1+\alpha_3+|\beta|+\delta+2)y} \end{vmatrix} & 1 & \begin{vmatrix} 1 & \frac{1}{(\alpha_2+\beta_1+\delta+1)_q} \\ 1 & \frac{1}{(\alpha_2+\beta_2+\delta+1)_q} \end{vmatrix} \\ q^{\alpha_3 x} \begin{vmatrix} q^{\gamma x} & q^{(\alpha_1+\alpha_2+|\beta|+\gamma+\delta+2)y} \\ 1 & q^{(\alpha_1+\alpha_2+|\beta|+\delta+2)y} \end{vmatrix} & 1 & \begin{vmatrix} 1 & \frac{1}{(\alpha_3+\beta_1+\delta+1)_q} \\ 1 & \frac{1}{(\alpha_3+\beta_2+\delta+1)_q} \end{vmatrix} \end{vmatrix}.$$

By taking the sum on y, this becomes

$$C_0 \sum_{0\le x} \begin{bmatrix} x+f \\ f \end{bmatrix} q^{(|\alpha|+|\beta|+\gamma+\delta+3)x} \begin{Vmatrix} 1 & \frac{1}{(\alpha_2+\alpha_3+|\beta|+\gamma+\delta+2)_q} & 1 & \begin{vmatrix} 1 & \frac{1}{(\alpha_1+\beta_1+\delta+1)_q} \\ 1 & \frac{1}{(\alpha_1+\beta_2+\delta+1)_q} \end{vmatrix} \\ 1 & \frac{1}{(\alpha_2+\alpha_3+|\beta|+\delta+2)_q} & & \\ 1 & \frac{1}{(\alpha_1+\alpha_3+|\beta|+\gamma+\delta+2)_q} & 1 & \begin{vmatrix} 1 & \frac{1}{(\alpha_2+\beta_1+\delta+1)_q} \\ 1 & \frac{1}{(\alpha_2+\beta_2+\delta+1)_q} \end{vmatrix} \\ 1 & \frac{1}{(\alpha_1+\alpha_3+|\beta|+\delta+2)_q} & & \\ 1 & \frac{1}{(\alpha_1+\alpha_2+|\beta|+\gamma+\delta+2)_q} & 1 & \begin{vmatrix} 1 & \frac{1}{(\alpha_3+\beta_1+\delta+1)_q} \\ 1 & \frac{1}{(\alpha_3+\beta_2+\delta+1)_q} \end{vmatrix} \\ 1 & \frac{1}{(\alpha_1+\alpha_2+|\beta|+\delta+2)_q} & & \end{Vmatrix}.$$

Substituting the equation $\sum_x \begin{bmatrix} x+f \\ f \end{bmatrix} q^{(|\alpha|+|\beta|+\gamma+\delta+3)x} = \frac{(q;q)_{|\alpha|+|\beta|+\gamma+\delta+2}}{(q;q)_{|\alpha|+|\beta|+\gamma+\delta+f+3}}$ into this formula, we obtain the above formula is equal to

$$C_1 \left\| \begin{matrix} 1 & \frac{1}{(\alpha_2+\alpha_3+|\beta|+\gamma+\delta+2)_q} \\ 1 & \frac{1}{(\alpha_2+\alpha_3+|\beta|+\delta+2)_q} \\ 1 & \frac{1}{(\alpha_1+\alpha_3+|\beta|+\gamma+\delta+2)_q} \\ 1 & \frac{1}{(\alpha_1+\alpha_3+|\beta|+\delta+2)_q} \\ 1 & \frac{1}{(\alpha_1+\alpha_2+|\beta|+\gamma+\delta+2)_q} \\ 1 & \frac{1}{(\alpha_1+\alpha_2+|\beta|+\delta+2)_q} \end{matrix} \right| \begin{matrix} 1 \\ \\ 1 \\ \\ 1 \\ \end{matrix} \left| \begin{matrix} 1 & \frac{1}{(\alpha_1+\beta_1+\delta+1)_q} \\ 1 & \frac{1}{(\alpha_1+\beta_2+\delta+1)_q} \\ 1 & \frac{1}{(\alpha_2+\beta_1+\delta+1)_q} \\ 1 & \frac{1}{(\alpha_2+\beta_2+\delta+1)_q} \\ 1 & \frac{1}{(\alpha_3+\beta_1+\delta+1)_q} \\ 1 & \frac{1}{(\alpha_3+\beta_2+\delta+1)_q} \end{matrix} \right\|,$$

where

$$C_1 = C_0 \frac{(q;q)_{|\alpha|+|\beta|+\gamma+\delta+2}}{(q;q)_{|\alpha|+|\beta|+\gamma+\delta+f+3}}.$$

By (16) we see the above identity equals

$$C_2 \begin{vmatrix} \frac{q^{\alpha_2}(\alpha_2+|\alpha|+2|\beta|+\gamma+2\delta+4)_q}{(\alpha_2+\alpha_3+|\beta|+\gamma+\delta+2)_q(\alpha_2+\alpha_3+|\beta|+\delta+2)_q} & \frac{(\alpha_1+\alpha_3+|\beta|+2\delta+2)_q}{(\alpha_1+\beta_1+\delta+1)_q(\alpha_1+\beta_2+\delta+1)_q} \\ \frac{q^{\alpha_1}(\alpha_1+|\alpha|+2|\beta|+\gamma+2\delta+4)_q}{(\alpha_1+\alpha_3+|\beta|+\gamma+\delta+2)_q(\alpha_1+\alpha_3+|\beta|+\delta+2)_q} & \frac{(\alpha_2+\alpha_3+|\beta|+2\delta+2)_q}{(\alpha_2+\beta_1+\delta+1)_q(\alpha_2+\beta_2+\delta+1)_q} \end{vmatrix},$$

where

$$C_2 = C_1 \frac{q^{|\beta|+2\delta+3}(q^{\alpha_3}-q^{\alpha_1})(q^{\alpha_3}-q^{\alpha_2})(q^{\beta_2}-q^{\beta_1})(\gamma)_q}{(\alpha_1+\alpha_2+|\beta|+\gamma+\delta+2)_q(\alpha_1+\alpha_2+|\beta|+\delta+2)_q(\alpha_3+\beta_1+\delta+1)_q(\alpha_3+\beta_2+\delta+1)_q}$$

If $\gamma = \delta$, then the above determinant becomes

$$\frac{q^{\alpha_2}(\alpha_2+|\alpha|+2|\beta|+3\gamma+4)_q}{(\alpha_2+\alpha_3+|\beta|+\gamma+2)_q(\alpha_2+\beta_1+\gamma+1)_q(\alpha_2+\beta_2+\gamma+1)_q} - \frac{q^{\alpha_1}(\alpha_1+|\alpha|+2|\beta|+3\gamma+4)_q}{(\alpha_1+\alpha_3+|\beta|+\gamma+2)_q(\alpha_1+\beta_1+\gamma+1)_q(\alpha_1+\beta_2+\gamma+1)_q}.$$

Applying (10), we see that the numerator of this difference factors as

$$(q^{\alpha_2}-q^{\alpha_1})(\alpha_1+\alpha_2+|\beta|+2\gamma+2)_q(|\alpha|+|\beta|+\beta_1+2\gamma+3)_q(|\alpha|+|\beta|+\beta_2+2\gamma+3)_q.$$

The reader may see that this equation also factors in the case that $\delta = 0$ and γ is arbitrary, but we don't treat this case here since it is included in the Insets case. This shows that consequently the above generating function becomes as

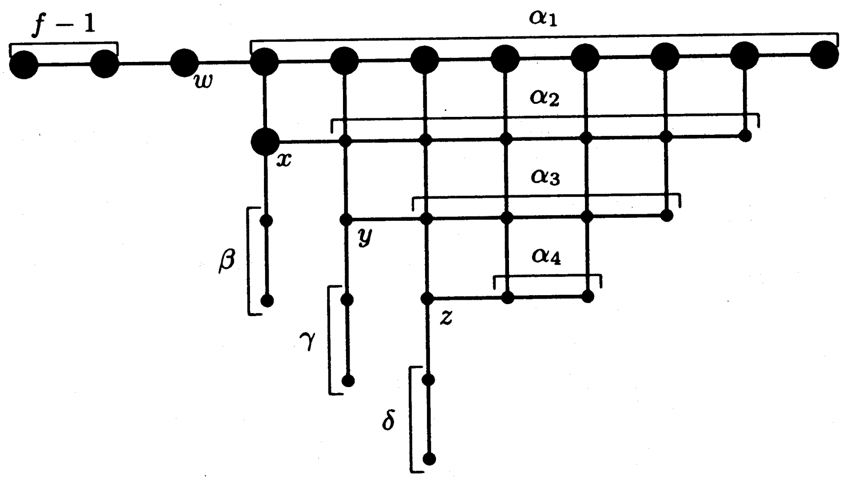
$$\frac{q^{\sum_j \binom{\beta_j+1}{2}+\binom{\gamma+1}{2}+\binom{\gamma+3}{2}+|\beta|}}{\prod_i (q;q)_{\alpha_i} \prod_j (q;q)_{\beta_j} (q;q)_\gamma (q;q)_{\gamma-1}} \cdot \frac{(q;q)_{|\alpha|+|\beta|+\gamma+\delta+2}}{(q;q)_{|\alpha|+|\beta|+\gamma+\delta+f+3}}$$
$$\times \frac{\prod_{1\le i\le j\le 3}(q^{\alpha_j}-q^{\alpha_i})(q^{\beta_2}-q^{\beta_1})\prod_{i=1}^{2}(\beta_i+|\alpha|+|\beta|+2\gamma+3)_q}{\prod_{i=1}^{3}(|\alpha|-\alpha_i+|\beta|+\gamma+2)_q \prod_{i=1}^{3}\prod_{j=1}^{2}(\alpha_i+\beta_j+\gamma+1)_q}.$$

This proves the theorem. □

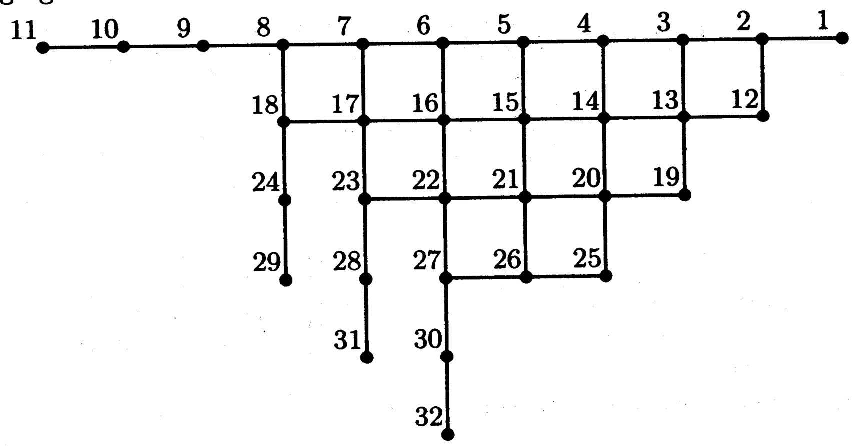
6.6 Banners

Next we consider Banners case. Let $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ satisfy $\alpha_1 > \alpha_2 > \alpha_3 > \alpha_4 \geq 0$. Let $f \geq 2$, β, γ and δ be nonnegative integers. Under these conditions let $P = P(\alpha, \beta, \gamma, \delta, f; 5)$ denote the poset defined by the following diagram, which will be called the Banners. The *top tree posets* is the filter which consists of the large solid dots as before.

Banners $(f \geq \gamma,\ \alpha_1 > \alpha_2 > \alpha_3 \geq 0,\ \beta, \gamma, \delta \geq 0)$



We consider a column-strict labeling ω_c, which is designated by the following figure.



Theorem 6.9 *The generating function $F(q) = F(P, \omega_c; q)$ of the (P, ω_c)-partitions is given by*

$$F(q) = \frac{q^{2\binom{\beta+2}{2}+\binom{\gamma+2}{2}+2\beta+\gamma+3}(q;q)_{|\alpha|+2\beta+\gamma+3}}{\prod_{i=1}^{4}(q;q)_{\alpha_i}(q;q)_{\beta}^{2}(q;q)_{\gamma}(q;q)_{|\alpha|+2\beta+\gamma+f+3}}.$$
$$\times \frac{(|\alpha|+2\beta+2\gamma+4)_q \prod_{1\leq i<j\leq 4}(q^{\alpha_j}-q^{\alpha_i})}{\prod_{i=1}^{4}(\alpha_i+\beta+1)_q \prod_{1\leq i<j\leq 4}(\alpha_i+\alpha_j+\beta+\gamma+2)_q}$$

Proof. By Lemma 4.1 the generating function $F(q) = F(P, \omega_c; q)$ of (P, ω_c)-partitions is given by

$$C_0 \sum_{0\leq w<x<y<z} q^{w+(\beta+1)x+(\gamma+1)y+(\delta+1)z} \begin{bmatrix} w+f-1 \\ f-1 \end{bmatrix} \begin{vmatrix} q^{\alpha_1 w} & q^{\alpha_1 x} & q^{\alpha_1 y} & q^{\alpha_1 z} \\ q^{\alpha_2 w} & q^{\alpha_2 x} & q^{\alpha_2 y} & q^{\alpha_2 z} \\ q^{\alpha_3 w} & q^{\alpha_3 x} & q^{\alpha_3 y} & q^{\alpha_3 z} \\ q^{\alpha_4 w} & q^{\alpha_4 x} & q^{\alpha_4 y} & q^{\alpha_4 z} \end{vmatrix};$$

where

$$C_0 = \frac{q^{\binom{\beta+1}{2}+\binom{\gamma+1}{2}+\binom{\delta+1}{2}}}{\prod_{i=1}^{4}(q;q)_{\alpha_i}(q;q)_{\beta}(q;q)_{\gamma}(q;q)_{\delta}}.$$

Before we continue our proof, we need two lemmas.

Lemma 6.10 *Let $a \geq 0$ be nonnegative integer and β, γ, δ be as above. Let*

$$G(q) = \sum_{0 \leq w < x < y < z} q^{(a+1)w+(\beta+1)x+(\gamma+1)y+(\delta+1)z} \begin{vmatrix} q^{\alpha_1 w} & q^{\alpha_1 x} & q^{\alpha_1 y} & q^{\alpha_1 z} \\ q^{\alpha_2 w} & q^{\alpha_2 x} & q^{\alpha_2 y} & q^{\alpha_2 z} \\ q^{\alpha_3 w} & q^{\alpha_3 x} & q^{\alpha_3 y} & q^{\alpha_3 z} \\ q^{\alpha_4 w} & q^{\alpha_4 x} & q^{\alpha_4 y} & q^{\alpha_4 z} \end{vmatrix}.$$

Then the above generating function $F(q)$ is given by

$$F(q) = C_0 \frac{(q;q)_{|\alpha|+\beta+\gamma+\delta+3}}{(q;q)_{|\alpha|+\beta+\gamma+\delta+f+3}} (a + |\alpha| + \beta + \gamma + \delta + 4)_q G(q)$$

Proof. In the equation

$$F(q) = C_0 \sum_{0 \leq w < x < y < z} q^{(|\alpha|+1)w+(\beta+1)x+(\gamma+1)y+(\delta+1)z} \begin{bmatrix} w+f-1 \\ f-1 \end{bmatrix}$$
$$\times \begin{vmatrix} 1 & q^{\alpha_1(x-w)} & q^{\alpha_1(y-w)} & q^{\alpha_1(z-w)} \\ 1 & q^{\alpha_2(x-w)} & q^{\alpha_2(y-w)} & q^{\alpha_2(z-w)} \\ 1 & q^{\alpha_3(x-w)} & q^{\alpha_3(y-w)} & q^{\alpha_3(z-w)} \\ 1 & q^{\alpha_4(x-w)} & q^{\alpha_4(y-w)} & q^{\alpha_4(z-w)} \end{vmatrix},$$

we put $s = x - w$, $t = y - w$ and $u = z - w$, then it becomes

$$\begin{aligned} F(q) &= C_0 \varphi(q) \sum_{0 \leq w} q^{(|\alpha|+\beta+\gamma+\delta+4)w} \begin{bmatrix} w+f-1 \\ f-1 \end{bmatrix} \\ &= C_0 \frac{(q;q)_{|\alpha|+\beta+\gamma+\delta+3}}{(q;q)_{|\alpha|+\beta+\gamma+\delta+f+3}} \varphi(q) \end{aligned}$$

where

$$\varphi(q) = \sum_{0 \leq s < t < u} q^{(\beta+1)s+(\gamma+1)t+(\delta+1)u} \begin{vmatrix} 1 & q^{\alpha_1 s} & q^{\alpha_1 t} & q^{\alpha_1 u} \\ 1 & q^{\alpha_2 s} & q^{\alpha_2 t} & q^{\alpha_2 u} \\ 1 & q^{\alpha_3 s} & q^{\alpha_3 t} & q^{\alpha_3 u} \\ 1 & q^{\alpha_4 s} & q^{\alpha_4 t} & q^{\alpha_4 u} \end{vmatrix},$$

Meanwhile, the same method shows that

$$G(q) = \frac{\varphi(q)}{(a + |\alpha| + \beta + \gamma + \delta + 4)_q}$$

This prove the lemma. □

Lemma 6.11 *Let $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4), \beta, \gamma, \delta, a$, and $G(q)$ be as above. If $a = \gamma$ and $\beta = \delta$, then*

$$G(q) = \mathrm{pf}\left[\frac{q^{\beta+1}(q^{\alpha_j} - q^{\alpha_i})}{(\alpha_i + \beta + 1)_q (\alpha_j + \beta + 1)_q (\alpha_i + \alpha_j + \beta + \gamma + 2)_q}\right]_{1 \leq i < j \leq 4}$$

Proof. If $a = \gamma$ and $\beta = \delta$ then $G(q)$ becomes

$$\sum_{0 \leq w < x < y < z} \mathrm{pf} \begin{bmatrix} 0 & q^{(\gamma+1)w+(\beta+1)x} & q^{(\gamma+1)w+(\beta+1)y} & q^{(\gamma+1)w+(\beta+1)z} \\ -q^{(\gamma+1)w+(\beta+1)x} & 0 & q^{(\gamma+1)x+(\beta+1)y} & q^{(\gamma+1)x+(\beta+1)z} \\ -q^{(\gamma+1)w+(\beta+1)y} & -q^{(\gamma+1)x+(\beta+1)y} & 0 & q^{(\gamma+1)y+(\beta+1)z} \\ -q^{(\gamma+1)w+(\beta+1)z} & -q^{(\gamma+1)x+(\beta+1)z} & -q^{(\gamma+1)y+(\beta+1)z} & 0 \end{bmatrix}$$
$$\times \begin{vmatrix} q^{\alpha_1 w} & q^{\alpha_1 x} & q^{\alpha_1 y} & q^{\alpha_1 z} \\ q^{\alpha_2 w} & q^{\alpha_2 x} & q^{\alpha_2 y} & q^{\alpha_2 z} \\ q^{\alpha_3 w} & q^{\alpha_3 x} & q^{\alpha_3 y} & q^{\alpha_3 z} \\ q^{\alpha_4 w} & q^{\alpha_4 x} & q^{\alpha_4 y} & q^{\alpha_4 z} \end{vmatrix}.$$

By Theorem 4.2 this sum is expressed by the Pfiffian pf $[Q_{ij}]_{1\le i<j\le 4}$ with

$$Q_{ij} = \sum_{0\le x<y} q^{(\gamma+1)x+(\beta+1)y} \begin{vmatrix} q^{\alpha_i x} & q^{\alpha_i y} \\ q^{\alpha_j x} & q^{\alpha_j y} \end{vmatrix}.$$

A simple direct calculation shows

$$Q_{ij} = \frac{q^{\beta+1}(q^{\alpha_j}-q^{\alpha_i})}{(\alpha_i+\beta+1)_q(\alpha_j+\beta+1)_q(\alpha_i+\alpha_j+\beta+\gamma+2)_q}$$

Proof of Theorem 6.9. By Lemma 6.11, we have

$$G(q) = \frac{q^{2\beta+2}}{\prod_{i=1}^4(\alpha_i+\beta+1)_q}\,\mathrm{pf}\left[\frac{q^{\alpha_j}-q^{\alpha_i}}{1-q^{\beta+\gamma+2}q^{\alpha_i+\alpha_j}}\right]_{1\le i<j\le 4}$$

Since $\mathrm{pf}\left[\frac{q^{\alpha_j}-q^{\alpha_i}}{1-q^{\beta+\gamma+2}q^{\alpha_i+\alpha_j}}\right]_{1\le i<j\le 4} = q^{-(\beta+\gamma+2)}\mathrm{pf}\left[\frac{q^{\alpha_j+\frac{\beta+\gamma+2}{2}}-q^{\alpha_i+\frac{\beta+\gamma+2}{2}}}{1-q^{\alpha_i+\frac{\beta+\gamma+2}{2}}q^{\alpha_j+\frac{\beta+\gamma+2}{2}}}\right]_{1\le i<j\le 4}$,
Lemma 6.3 implies

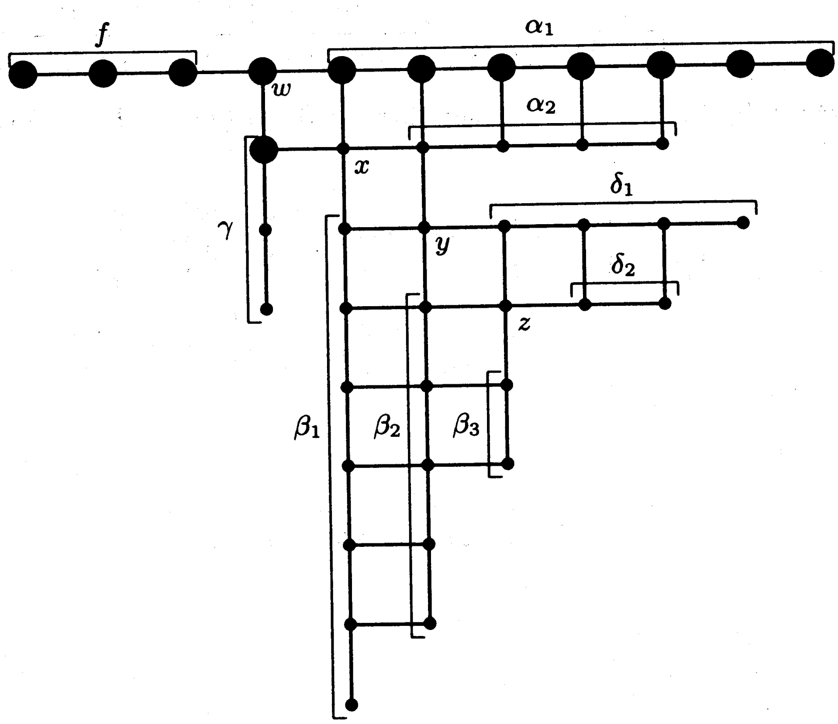
$$G(q) = \frac{q^{4\beta+2\gamma+6}}{\prod_{i=1}^4(\alpha_i+\beta+1)_q}\prod_{1\le i<j\le 4}\frac{q^{\alpha_j}-q^{\alpha_i}}{(\alpha_i+\alpha_j+\beta+\gamma+2)_q}$$

This and Lemma 6.10 immediately implies the theorem. □

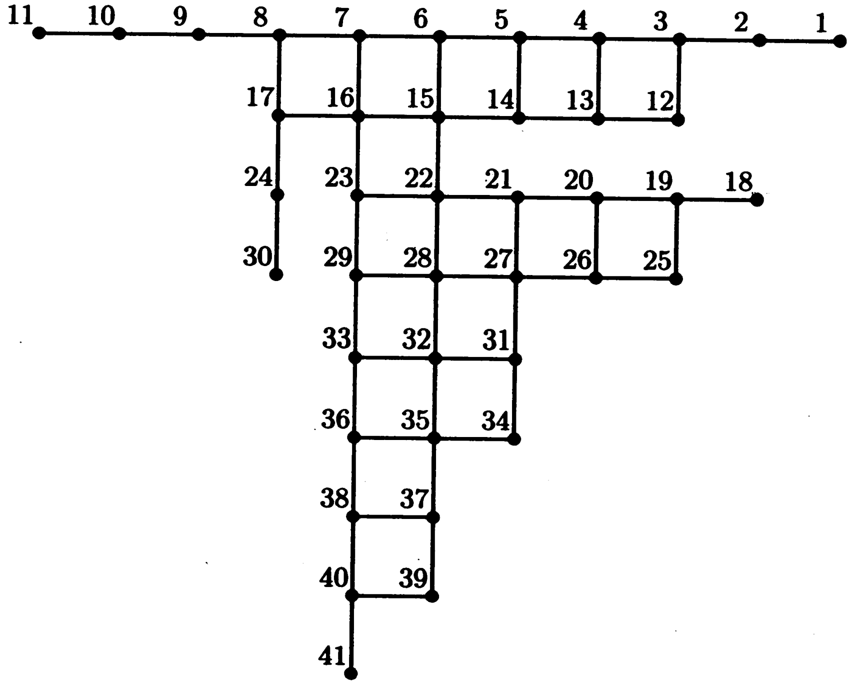
6.7 Nooks

Next we consider Nooks case. Let $\alpha=(\alpha_1,\alpha_2)$ satisfy $\alpha_1>\alpha_2>0$, $\beta=(\beta_1,\beta_2,\beta_3)$ satisfy $\beta_1>\beta_2>\beta_3\ge 0$, and $\delta=(\delta_1,\delta_2)$ satisfy $\delta_1>\delta_2\ge 0$. Let $f\ge 2$ and $\gamma>0$ be positive integers. Under these conditions let $P=P(\alpha,\beta,f,\gamma,\delta;5)$ denote the poset defined by the following diagram, which will be called the Nooks. The *top tree posets* is the filter which consists of the large solid dots as before.

Nooks $(f\ge\gamma,\ \alpha_1>\alpha_2\ge 0,\ \beta_1>\beta_2>\beta_3\ge 0)$



First we consider a column-strict labeling which will be realized as, for example, the following labels.



Theorem 6.12 *If $\delta_2 = 0$ and $\gamma = \delta_1$, then the generating function of the above (P, ω_c)-partitons is given by*

$$F(P,\omega_c;q) = \frac{q^{\sum_j \binom{\beta_j+2}{2}+\binom{\gamma+3}{2}}}{\prod_{i=1}^{2}(q;q)_{\alpha_i-1}\prod_{j=1}^{3}(q;q)_{\beta_j}(q;q)_{\gamma-1}^{2}} \frac{(q;q)_{|\alpha|+|\beta|+\gamma+|\delta|+3}}{(q;q)_{|\alpha|+|\beta|+\gamma+|\delta|+f+4}}$$
$$\times \frac{(q^{\alpha_2}-q^{\alpha_1})\prod_{1\le i<j\le 3}(q^{\beta_j}-q^{\beta_i})}{(|\alpha|+|\beta|+2\gamma+3)_q(|\alpha|+|\beta|+\gamma+3)_q}$$
$$\times \frac{\prod_{j=1}^{3}(|\alpha|+|\beta|+\beta_j+2\gamma+4)_q}{\prod_{i=1}^{2}\prod_{j=1}^{3}(\alpha_i+|\beta|-\beta_j+\gamma+2)_q\prod_{j=1}^{3}(\beta_j+\gamma+1)_q\prod_{j=1}^{3}(\beta_j+1)_q}$$

Proof. By Lemma 4.1 the generating function $F(q) = F(P,\omega_c;q)$ is given by

$$C_0 \sum_{0\le w\le x\le y\le z} \begin{bmatrix} w+f \\ f \end{bmatrix} q^{w+x+y+z} \begin{vmatrix} q^{\gamma w} & q^{\gamma x} \\ 1 & 1 \end{vmatrix} \cdot \begin{vmatrix} q^{\alpha_1 w} & q^{\alpha_1 x} & q^{\alpha_1 y} \\ q^{\alpha_2 w} & q^{\alpha_2 x} & q^{\alpha_2 y} \\ q^{\alpha_3 w} & q^{\alpha_3 x} & q^{\alpha_3 y} \end{vmatrix}$$
$$\times \begin{vmatrix} q^{\beta_1 x} & q^{\beta_1 y} & q^{\beta_1 z} \\ q^{\beta_2 x} & q^{\beta_2 y} & q^{\beta_2 z} \\ q^{\beta_3 x} & q^{\beta_3 y} & q^{\beta_3 z} \end{vmatrix} \cdot \begin{vmatrix} q^{\delta_1 y} & q^{\delta_1 z} \\ q^{\delta_2 y} & q^{\delta_2 z} \end{vmatrix},$$

where

$$C_0 = \frac{q^{\sum_j \binom{\beta_j+1}{2}+\binom{\gamma+1}{2}}}{\prod_{i=1}^{3}(q;q)_{\alpha_i}\prod_{j=1}^{3}(q;q)_{\beta_i}(q;q)_{\gamma}\prod_{k=1}^{2}(q;q)_{\delta_k}}.$$

By taking sum on z, it becomes

$$C_0 \sum_{0\le w\le x\le y} \begin{bmatrix} w+f \\ f \end{bmatrix} q^{w+x+y} \begin{vmatrix} q^{\gamma w} & q^{\gamma x} \\ 1 & 1 \end{vmatrix} \cdot \begin{vmatrix} q^{\alpha_1 w} & q^{\alpha_1 x} & q^{\alpha_1 y} \\ q^{\alpha_2 w} & q^{\alpha_2 x} & q^{\alpha_2 y} \\ q^{\alpha_3 w} & q^{\alpha_3 x} & q^{\alpha_3 y} \end{vmatrix}$$

$$\times \begin{vmatrix} q^{\beta_1 x} & q^{\beta_1 y} & \begin{vmatrix} q^{\delta_1 y} & \frac{q^{(\beta_1+\delta_1+1)y}}{(\beta_1+\delta_1+1)_q} \\ q^{\delta_2 y} & \frac{q^{(\beta_1+\delta_2+1)y}}{(\beta_1+\delta_2+1)_q} \end{vmatrix} \\ q^{\beta_2 x} & q^{\beta_2 y} & \begin{vmatrix} q^{\delta_1 y} & \frac{q^{(\beta_2+\delta_1+1)y}}{(\beta_2+\delta_1+1)_q} \\ q^{\delta_2 y} & \frac{q^{(\beta_2+\delta_2+1)y}}{(\beta_2+\delta_2+1)_q} \end{vmatrix} \\ q^{\beta_3 x} & q^{\beta_3 y} & \begin{vmatrix} q^{\delta_1 y} & \frac{q^{(\beta_3+\delta_1+1)y}}{(\beta_3+\delta_1+1)_q} \\ q^{\delta_2 y} & \frac{q^{(\beta_3+\delta_2+1)y}}{(\beta_3+\delta_2+1)_q} \end{vmatrix} \end{vmatrix},$$

By expanding the last determinant along the first column, we see that this equals

$$C_0 \sum_{0\le w\le x\le y} \begin{bmatrix} w+f \\ f \end{bmatrix} q^{w+x+y} \begin{vmatrix} q^{\gamma w} & q^{\gamma x} \\ 1 & 1 \end{vmatrix} \cdot \begin{vmatrix} q^{\alpha_1 w} & q^{\alpha_1 x} & q^{\alpha_1 y} \\ q^{\alpha_2 w} & q^{\alpha_2 x} & q^{\alpha_2 y} \\ q^{\alpha_3 w} & q^{\alpha_3 x} & q^{\alpha_3 y} \end{vmatrix}$$

$$\times \begin{vmatrix} q^{\beta_1 x+(\beta_2+\beta_3+|\delta|+1)y} & 1 & \begin{vmatrix} 1 & \frac{1}{(\beta_1+\delta_1+1)_q} \\ 1 & \frac{1}{(\beta_1+\delta_2+1)_q} \end{vmatrix} \\ q^{\beta_2 x+(\beta_1+\beta_3+|\delta|+1)y} & 1 & \begin{vmatrix} 1 & \frac{1}{(\beta_2+\delta_1+1)_q} \\ 1 & \frac{1}{(\beta_2+\delta_2+1)_q} \end{vmatrix} \\ q^{\beta_3 x+(\beta_1+\beta_2+|\delta|+1)y} & 1 & \begin{vmatrix} 1 & \frac{1}{(\beta_3+\delta_1+1)_q} \\ 1 & \frac{1}{(\beta_3+\delta_2+1)_q} \end{vmatrix} \end{vmatrix},$$

By (11) this becomes

$$C_1 \sum_{0\le w\le x\le y} \begin{bmatrix} w+f \\ f \end{bmatrix} q^{w+x+y} \begin{vmatrix} q^{\gamma w} & q^{\gamma x} \\ 1 & 1 \end{vmatrix} \cdot \begin{vmatrix} q^{\alpha_1 w} & q^{\alpha_1 x} & q^{\alpha_1 y} \\ q^{\alpha_2 w} & q^{\alpha_2 x} & q^{\alpha_2 y} \\ q^{\alpha_3 w} & q^{\alpha_3 x} & q^{\alpha_3 y} \end{vmatrix}$$

$$\times \begin{vmatrix} q^{\beta_1 x+(\beta_2+\beta_3+|\delta|+1)y} & 1 & \frac{q^{\beta_1}}{(\beta_1+\delta_1+1)_q(\beta_1+\delta_2+1)_q} \\ q^{\beta_2 x+(\beta_1+\beta_3+|\delta|+1)y} & 1 & \frac{q^{\beta_2}}{(\beta_2+\delta_1+1)_q(\beta_2+\delta_2+1)_q} \\ q^{\beta_3 x+(\beta_1+\beta_2+|\delta|+1)y} & 1 & \frac{q^{\beta_3}}{(\beta_3+\delta_1+1)_q(\beta_3+\delta_2+1)_q} \end{vmatrix},$$

where $C_1 = q(q^{\delta_2} - q^{\delta_1})C_0$. Now we put the last determinant into the last second one, then we get

$$C_1 \sum_{0\le w\le x\le y} \begin{bmatrix} w+f \\ f \end{bmatrix} q^{w+x} \begin{vmatrix} q^{\gamma w} & q^{\gamma x} \\ 1 & 1 \end{vmatrix}$$
$$\times \begin{vmatrix} q^{\alpha_1 w} & q^{\alpha_1 x} & \begin{vmatrix} q^{\beta_1 x+(\alpha_1+\beta_2+\beta_3+|\delta|+2)y} & 1 & \frac{q^{\beta_1}}{(\beta_1+\delta_1+1)_q(\beta_1+\delta_2+1)_q} \\ q^{\beta_2 x+(\alpha_1+\beta_1+\beta_3+|\delta|+2)y} & 1 & \frac{q^{\beta_2}}{(\beta_2+\delta_1+1)_q(\beta_2+\delta_2+1)_q} \\ q^{\beta_3 x+(\alpha_1+\beta_1+\beta_2+|\delta|+2)y} & 1 & \frac{q^{\beta_3}}{(\beta_3+\delta_1+1)_q(\beta_3+\delta_2+1)_q} \end{vmatrix} \\ q^{\alpha_2 w} & q^{\alpha_2 x} & \begin{vmatrix} q^{\beta_1 x+(\alpha_2+\beta_2+\beta_3+|\delta|+2)y} & 1 & \frac{q^{\beta_1}}{(\beta_1+\delta_1+1)_q(\beta_1+\delta_2+1)_q} \\ q^{\beta_2 x+(\alpha_2+\beta_1+\beta_3+|\delta|+2)y} & 1 & \frac{q^{\beta_2}}{(\beta_2+\delta_1+1)_q(\beta_2+\delta_2+1)_q} \\ q^{\beta_3 x+(\alpha_2+\beta_1+\beta_2+|\delta|+2)y} & 1 & \frac{q^{\beta_3}}{(\beta_3+\delta_1+1)_q(\beta_3+\delta_2+1)_q} \end{vmatrix} \\ 1 & 1 & \begin{vmatrix} q^{\beta_1 x+(\alpha_3+\beta_2+\beta_3+|\delta|+2)y} & 1 & \frac{q^{\beta_1}}{(\beta_1+\delta_1+1)_q(\beta_1+\delta_2+1)_q} \\ q^{\beta_2 x+(\alpha_3+\beta_1+\beta_3+|\delta|+2)y} & 1 & \frac{q^{\beta_2}}{(\beta_2+\delta_1+1)_q(\beta_2+\delta_2+1)_q} \\ q^{\beta_3 x+(\alpha_3+\beta_1+\beta_2+|\delta|+2)y} & 1 & \frac{q^{\beta_3}}{(\beta_3+\delta_1+1)_q(\beta_3+\delta_2+1)_q} \end{vmatrix} \end{vmatrix}.$$

Taking the sum on y, we obtain

$$C_1 \sum_{0\le w\le x} \begin{bmatrix} w+f \\ f \end{bmatrix} q^{w+x} \begin{vmatrix} q^{\gamma w} & q^{\gamma x} \\ 1 & 1 \end{vmatrix}$$
$$\times \begin{vmatrix} q^{\alpha_1 w} & q^{\alpha_1 x} & q^{(\alpha_1+|\beta|+|\delta|+2)x} \begin{vmatrix} \frac{1}{(\alpha_1+\beta_2+\beta_3+|\delta|+2)_q} & 1 & \frac{q^{\beta_1}}{(\beta_1+\delta_1+1)_q(\beta_1+\delta_2+1)_q} \\ \frac{1}{(\alpha_1+\beta_1+\beta_3+|\delta|+2)_q} & 1 & \frac{q^{\beta_2}}{(\beta_2+\delta_1+1)_q(\beta_2+\delta_2+1)_q} \\ \frac{1}{(\alpha_1+\beta_1+\beta_2+|\delta|+2)_q} & 1 & \frac{q^{\beta_3}}{(\beta_3+\delta_1+1)_q(\beta_3+\delta_2+1)_q} \end{vmatrix} \\ q^{\alpha_2 w} & q^{\alpha_2 x} & q^{(\alpha_2+|\beta|+|\delta|+2)x} \begin{vmatrix} \frac{1}{(\alpha_2+\beta_2+\beta_3+|\delta|+2)_q} & 1 & \frac{q^{\beta_1}}{(\beta_1+\delta_1+1)_q(\beta_1+\delta_2+1)_q} \\ \frac{1}{(\alpha_2+\beta_1+\beta_3+|\delta|+2)_q} & 1 & \frac{q^{\beta_2}}{(\beta_2+\delta_1+1)_q(\beta_2+\delta_2+1)_q} \\ \frac{1}{(\alpha_2+\beta_1+\beta_2+|\delta|+2)_q} & 1 & \frac{q^{\beta_3}}{(\beta_3+\delta_1+1)_q(\beta_3+\delta_2+1)_q} \end{vmatrix} \\ q^{\alpha_3 w} & q^{\alpha_3 x} & q^{(\alpha_3+|\beta|+|\delta|+2)x} \begin{vmatrix} \frac{1}{(\alpha_3+\beta_2+\beta_3+|\delta|+2)_q} & 1 & \frac{q^{\beta_1}}{(\beta_1+\delta_1+1)_q(\beta_1+\delta_2+1)_q} \\ \frac{1}{(\alpha_3+\beta_1+\beta_3+|\delta|+2)_q} & 1 & \frac{q^{\beta_2}}{(\beta_2+\delta_1+1)_q(\beta_2+\delta_2+1)_q} \\ \frac{1}{(\alpha_3+\beta_1+\beta_2+|\delta|+2)_q} & 1 & \frac{q^{\beta_3}}{(\beta_3+\delta_1+1)_q(\beta_3+\delta_2+1)_q} \end{vmatrix} \end{vmatrix}.$$

Again, by expanding the last determinant along the first column, we see that it is equal to

$$C_1 \sum_{0\le w\le x} \begin{bmatrix} w+f \\ f \end{bmatrix} q^{w} \begin{vmatrix} q^{\alpha_1 w+(\alpha_2+\alpha_3+|\beta|+|\delta|+3)x} \begin{vmatrix} q^{\gamma w} & q^{\gamma x} \\ 1 & 1 \end{vmatrix} & 1 & G(\alpha_1;q) \\ q^{\alpha_2 w+(\alpha_1+\alpha_3+|\beta|+|\delta|+3)x} \begin{vmatrix} q^{\gamma w} & q^{\gamma x} \\ 1 & 1 \end{vmatrix} & 1 & G(\alpha_2;q) \\ q^{\alpha_3 w+(\alpha_1+\alpha_2+|\beta|+|\delta|+3)x} \begin{vmatrix} q^{\gamma w} & q^{\gamma x} \\ 1 & 1 \end{vmatrix} & 1 & G(\alpha_3;q) \end{vmatrix}.$$

Here we put

$$G(\alpha_i;q) = \begin{vmatrix} \frac{1}{(\alpha_i+\beta_2+\beta_3+|\delta|+2)_q} & 1 & \frac{q^{\beta_1}}{(\beta_1+\delta_1+1)_q(\beta_1+\delta_2+1)_q} \\ \frac{1}{(\alpha_i+\beta_1+\beta_3+|\delta|+2)_q} & 1 & \frac{q^{\beta_2}}{(\beta_2+\delta_1+1)_q(\beta_2+\delta_2+1)_q} \\ \frac{1}{(\alpha_i+\beta_1+\beta_2+|\delta|+2)_q} & 1 & \frac{q^{\beta_3}}{(\beta_3+\delta_1+1)_q(\beta_3+\delta_2+1)_q} \end{vmatrix}.$$

Now taking the sum on x it becomes

$$C_1 \sum_{0 \le w} \begin{bmatrix} w+f \\ f \end{bmatrix} q^{(|\alpha|+|\beta|+\gamma+|\delta|+4)w} \begin{vmatrix} \begin{vmatrix} 1 & \frac{1}{(\alpha_2+\alpha_3+|\beta|+\gamma+|\delta|+3)_q} \\ 1 & \frac{1}{(\alpha_2+\alpha_3+|\beta|+|\delta|+3)_q} \end{vmatrix} & 1 & G(\alpha_1;q) \\ \begin{vmatrix} 1 & \frac{1}{(\alpha_1+\alpha_3+|\beta|+\gamma+|\delta|+3)_q} \\ 1 & \frac{1}{(\alpha_1+\alpha_3+|\beta|+|\delta|+3)_q} \end{vmatrix} & 1 & G(\alpha_2;q) \\ \begin{vmatrix} 1 & \frac{1}{(\alpha_1+\alpha_2+|\beta|+\gamma+|\delta|+3)_q} \\ 1 & \frac{1}{(\alpha_1+\alpha_2+|\beta|+|\delta|+3)_q} \end{vmatrix} & 1 & G(\alpha_3;q) \end{vmatrix}.$$

Finally taking the sum on w, this becomes

$$C_1 \frac{(q;q)_{|\alpha|+|\beta|+\gamma+|\delta|+3}}{(q;q)_{|\alpha|+|\beta|+\gamma+|\delta|+f+4}} \begin{vmatrix} \begin{vmatrix} 1 & \frac{1}{(\alpha_2+\alpha_3+|\beta|+\gamma+|\delta|+3)_q} \\ 1 & \frac{1}{(\alpha_2+\alpha_3+|\beta|+|\delta|+3)_q} \end{vmatrix} & 1 & G(\alpha_1;q) \\ \begin{vmatrix} 1 & \frac{1}{(\alpha_1+\alpha_3+|\beta|+\gamma+|\delta|+3)_q} \\ 1 & \frac{1}{(\alpha_1+\alpha_3+|\beta|+|\delta|+3)_q} \end{vmatrix} & 1 & G(\alpha_2;q) \\ \begin{vmatrix} 1 & \frac{1}{(\alpha_1+\alpha_2+|\beta|+\gamma+|\delta|+3)_q} \\ 1 & \frac{1}{(\alpha_1+\alpha_2+|\beta|+|\delta|+3)_q} \end{vmatrix} & 1 & G(\alpha_3;q) \end{vmatrix}.$$

By (13) we have

$$G(\alpha_i;q) = \frac{(q^{\beta_3}-q^{\beta_1})(q^{\beta_3}-q^{\beta_2})q^{\alpha_i+|\delta|+2}}{(\alpha_i+\beta_1+\beta_2+|\delta|+2)_q(\beta_3+\delta_1+1)_q(\beta_3+\delta_2+1)_q} \times \begin{vmatrix} \frac{q^{\beta_2}}{(\alpha_i+\beta_2+\beta_3+|\delta|+2)_q} & \frac{(\beta_1+\beta_3+|\delta|+2)_q}{(\beta_1+\delta_1+1)_q(\beta_1+\delta_2+1)_q} \\ \frac{q^{\beta_1}}{(\alpha_i+\beta_1+\beta_3+|\delta|+2)_q} & \frac{(\beta_2+\beta_3+|\delta|+2)_q}{(\beta_2+\delta_1+1)_q(\beta_2+\delta_2+1)_q} \end{vmatrix}.$$

Substituting this identity into the above determinant, we see that $F(q)$ is equal to

$$C_2 \begin{vmatrix} \frac{q^{\alpha_2+\alpha_3}}{(\alpha_2+\alpha_3+|\beta|+\gamma+|\delta|+3)_q(\alpha_2+\alpha_3+|\beta|+|\delta|+3)_q} & 1 & \frac{q^{\alpha_1}}{(\alpha_1+\beta_1+\beta_2+|\delta|+2)_q} \begin{vmatrix} \frac{q^{\beta_2}}{(\alpha_1+\beta_2+\beta_3+|\delta|+2)_q} & \frac{(\beta_1+\beta_3+|\delta|+2)_q}{(\beta_1+\delta_1+1)_q(\beta_1+\delta_2+1)_q} \\ \frac{q^{\beta_1}}{(\alpha_1+\beta_1+\beta_3+|\delta|+2)_q} & \frac{(\beta_2+\beta_3+|\delta|+2)_q}{(\beta_2+\delta_1+1)_q(\beta_2+\delta_2+1)_q} \end{vmatrix} \\ \frac{q^{\alpha_1+\alpha_3}}{(\alpha_1+\alpha_3+|\beta|+\gamma+|\delta|+3)_q(\alpha_1+\alpha_3+|\beta|+|\delta|+3)_q} & 1 & \frac{q^{\alpha_2}}{(\alpha_2+\beta_1+\beta_2+|\delta|+2)_q} \begin{vmatrix} \frac{q^{\beta_2}}{(\alpha_2+\beta_2+\beta_3+|\delta|+2)_q} & \frac{(\beta_1+\beta_3+|\delta|+2)_q}{(\beta_1+\delta_1+1)_q(\beta_1+\delta_2+1)_q} \\ \frac{q^{\beta_1}}{(\alpha_2+\beta_1+\beta_3+|\delta|+2)_q} & \frac{(\beta_2+\beta_3+|\delta|+2)_q}{(\beta_2+\delta_1+1)_q(\beta_2+\delta_2+1)_q} \end{vmatrix} \\ \frac{q^{\alpha_1+\alpha_2}}{(\alpha_1+\alpha_2+|\beta|+\gamma+|\delta|+3)_q(\alpha_1+\alpha_2+|\beta|+|\delta|+3)_q} & 1 & \frac{q^{\alpha_3}}{(\alpha_3+\beta_1+\beta_2+|\delta|+2)_q} \begin{vmatrix} \frac{q^{\beta_2}}{(\alpha_3+\beta_2+\beta_3+|\delta|+2)_q} & \frac{(\beta_1+\beta_3+|\delta|+2)_q}{(\beta_1+\delta_1+1)_q(\beta_1+\delta_2+1)_q} \\ \frac{q^{\beta_1}}{(\alpha_3+\beta_1+\beta_3+|\delta|+2)_q} & \frac{(\beta_2+\beta_3+|\delta|+2)_q}{(\beta_2+\delta_1+1)_q(\beta_2+\delta_2+1)_q} \end{vmatrix} \end{vmatrix}$$

where

$$C_2 = C_1 q^{|\beta|+2|\delta|+5}(\gamma)_q \frac{(q;q)_{|\alpha|+|\beta|+\gamma+|\delta|+3}}{(q;q)_{|\alpha|+|\beta|+\gamma+|\delta|+f+4}} \frac{(q^{\beta_3}-q^{\beta_1})(q^{\beta_3}-q^{\beta_2})}{(\beta_3+\delta_1+1)_q(\beta_3+\delta_2+1)_q}.$$

Nextly if we substruct the bottom row from the first row, then substract the bottom row from the second row, and expand the determinant along the second column, we obtain

$$F(q) = \frac{C_3}{(\alpha_3+\beta_1+\beta_2+|\delta|+2)_q} \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix}$$

where, by (9),

$$A_{i1} = \frac{q^{\alpha_1+\alpha_2-\alpha_i}(q^{\alpha_3}-q^{\alpha_i})(\alpha_1+\alpha_2-\alpha_i+|\alpha|+2|\beta|+\gamma+2|\delta|+6)_q}{(|\alpha|-\alpha_i+|\beta|+\gamma+|\delta|+3)_q(|\alpha|-\alpha_i+|\beta|+|\delta|+3)_q},$$

$$A_{i2} = \frac{q^{\alpha_3} - q^{\alpha_i}}{(\alpha_i + \beta_1 + \beta_2 + |\delta| + 2)_q}$$
$$\times \begin{vmatrix} \frac{q^{\beta_2}(\alpha_i+\alpha_3+\beta_2+|\beta|+2|\delta|+4)_q}{(\alpha_i+\beta_2+\beta_3+|\delta|+2)_q(\alpha_3+\beta_2+\beta_3+|\delta|+2)_q} & \frac{(\beta_1+\beta_3+|\delta|+2)_q}{(\beta_1+\delta_1+1)_q(\beta_1+\delta_2+1)_q} \\ \frac{q^{\beta_1}(\alpha_i+\alpha_3+\beta_1+|\beta|+2|\delta|+4)_q}{(\alpha_i+\beta_1+\beta_3+|\delta|+2)_q(\alpha_3+\beta_1+\beta_3+|\delta|+2)_q} & \frac{(\beta_2+\beta_3+|\delta|+2)_q}{(\beta_2+\delta_1+1)_q(\beta_2+\delta_2+1)_q} \end{vmatrix},$$

and

$$C_3 = \frac{C_2}{(\alpha_1 + \alpha_2 + |\beta| + \gamma + |\delta| + 3)_q(\alpha_1 + \alpha_2 + |\beta| + |\delta| + 3)_q}.$$

Further, if $\alpha_3 = 0$, then A_{i2} becomes as

$$A_{i2} = \frac{(\alpha_i)_q}{(\alpha_i + \beta_1 + \beta_2 + |\delta| + 2)_q} \begin{vmatrix} \frac{q^{\beta_2}(\alpha_i+\beta_2+|\beta|+2|\delta|+4)_q}{(\alpha_i+\beta_2+\beta_3+|\delta|+2)_q} & \frac{1}{(\beta_1+\delta_1+1)_q(\beta_1+\delta_2+1)_q} \\ \frac{q^{\beta_1}(\alpha_i+\beta_1+|\beta|+2|\delta|+4)_q}{(\alpha_i+\beta_1+\beta_3+|\delta|+2)_q} & \frac{1}{(\beta_2+\delta_1+1)_q(\beta_2+\delta_2+1)_q} \end{vmatrix}$$
$$= \frac{(\alpha_i)_q}{(\alpha_i + \beta_1 + \beta_2 + |\delta| + 2)_q}$$
$$\times \frac{G}{\prod_{j=1}^{2}(\alpha_i + \beta_j + \beta_3 + |\delta| + 2)_q \prod_{j=1}^{2}\prod_{k=1}^{2}(\beta_j + \delta_k + 1)_q},$$

where

$$G = q^{\beta_2}(\alpha_i + \beta_2 + |\beta| + 2|\delta| + 4)_q(\alpha_i + \beta_1 + \beta_3 + |\delta| + 2)_q(\beta_1 + \delta_1 + 1)_q(\beta_1 + \delta_2 + 1)_q$$
$$- q^{\beta_1}(\alpha_i + \beta_1 + |\beta| + 2|\delta| + 4)_q(\alpha_i + \beta_2 + \beta_3 + |\delta| + 2)_q(\beta_2 + \delta_1 + 1)_q(\beta_2 + \delta_2 + 1)_q.$$

By (10) we have

$$G = (q^{\beta_2} - q^{\beta_1})(\beta_1 + \beta_2 + |\delta| + 2)_q(\alpha_i + |\beta| + \delta_1 + |\delta| + 3)_q(\alpha_i + |\beta| + \delta_2 + |\delta| + 3)_q.$$

This implies that $F(q)$ is equal to

$$C_4 \begin{vmatrix} \frac{q^{\alpha_2}(\alpha_2+|\alpha|+2|\beta|+\gamma+2|\delta|+6)_q}{(\alpha_2+|\beta|+\gamma+|\delta|+3)_q(\alpha_2+|\beta|+|\delta|+3)_q} & \frac{(\alpha_1+|\beta|+\delta_1+|\delta|+3)_q(\alpha_1+|\beta|+\delta_2+|\delta|+3)_q}{(\alpha_1+\beta_1+\beta_2+|\delta|+2)_q(\alpha_1+\beta_1+\beta_3+|\delta|+2)_q(\alpha_1+\beta_2+\beta_3+|\delta|+2)_q} \\ \frac{q^{\alpha_1}(\alpha_1+|\alpha|+2|\beta|+\gamma+2|\delta|+6)_q}{(\alpha_1+|\beta|+\gamma+|\delta|+3)_q(\alpha_1+|\beta|+|\delta|+3)_q} & \frac{(\alpha_2+|\beta|+\delta_1+|\delta|+3)_q(\alpha_2+|\beta|+\delta_2+|\delta|+3)_q}{(\alpha_2+\beta_1+\beta_2+|\delta|+2)_q(\alpha_2+\beta_1+\beta_3+|\delta|+2)_q(\alpha_2+\beta_2+\beta_3+|\delta|+2)_q} \end{vmatrix}$$

where

$$C_4 = C_3 \frac{(\alpha_1)_q(\alpha_2)_q(q^{\beta_2} - q^{\beta_1})}{\prod_{j=1}^{2}\prod_{k=1}^{2}(\beta_j + \delta_k + 1)_q}$$

Consequently, $F(q)$ becomes

$$C_0 q^{|\beta|+2|\delta|+6} \frac{(q;q)_{|\alpha|+|\beta|+\gamma+|\delta|+3}}{(q;q)_{|\alpha|+|\beta|+\gamma+|\delta|+f+4}} \frac{(\alpha_1)_q(\alpha_2)_q(\gamma)_q \prod_{1\le i<j\le 3}(q^{\beta_j} - q^{\beta_i})}{\prod_{j=1}^{3}\prod_{k=1}^{2}(\beta_j + \delta_k + 1)_q}$$
$$\times \frac{q^{\delta_2} - q^{\delta_1}}{(|\alpha| + |\beta| + \gamma + |\delta| + 3)_q(|\alpha| + |\beta| + |\delta| + 3)_q}$$
$$\times \frac{H}{\prod_{i=1}^{2}(\alpha_i + |\beta| + \gamma + |\delta| + 3)_q \prod_{i=1}^{2}(\alpha_i + |\beta| + |\delta| + 3)_q \prod_{i=1}^{2}\prod_{j=1}^{3}(\alpha_i + |\beta| - \beta_j + |\delta| + 2)_q},$$

where

$$H = q^{\alpha_2}(\alpha_2 + |\alpha| + 2|\beta| + \gamma + 2|\delta| + 6)_q(\alpha_2 + |\beta| + \delta_1 + |\delta| + 3)_q(\alpha_2 + |\beta| + \delta_2 + |\delta| + 3)_q$$
$$\times (\alpha_1 + |\beta| + \gamma + |\delta| + 3)_q(\alpha_1 + |\beta| + |\delta| + 3)_q$$
$$\times (\alpha_1 + \beta_1 + \beta_2 + |\delta| + 2)_q(\alpha_1 + \beta_1 + \beta_3 + |\delta| + 2)_q(\alpha_1 + \beta_2 + \beta_3 + |\delta| + 2)_q$$
$$- q^{\alpha_1}(\alpha_1 + |\alpha| + 2|\beta| + \gamma + 2|\delta| + 6)_q(\alpha_1 + |\beta| + \delta_1 + |\delta| + 3)_q(\alpha_1 + |\beta| + \delta_2 + |\delta| + 3)_q$$
$$\times (\alpha_2 + |\beta| + \gamma + |\delta| + 3)_q(\alpha_2 + |\beta| + |\delta| + 3)_q$$
$$\times (\alpha_2 + \beta_1 + \beta_2 + |\delta| + 2)_q(\alpha_2 + \beta_1 + \beta_3 + |\delta| + 2)_q(\alpha_2 + \beta_2 + \beta_3 + |\delta| + 2)_q.$$

Further, assume that $\delta_2 = 0$ and $\gamma = \delta_1 = |\delta|$. Then H becomes as

$$
\begin{aligned}
H =& \prod_{i=1}^{2}(\alpha_i + |\beta| + 2\gamma + 3)_q \prod_{i=1}^{2}(\alpha_i + |\beta| + \gamma + 3)_q \\
& \{q^{\alpha_2}(\alpha_2 + |\alpha| + 2|\beta| + 3\gamma + 6)_q(\alpha_1 + \beta_1 + \beta_2 + \gamma + 2)_q \\
& \times (\alpha_1 + \beta_1 + \beta_3 + \gamma + 2)_q(\alpha_1 + \beta_2 + \beta_3 + \gamma + 2)_q \\
& -q^{\alpha_1}(\alpha_1 + |\alpha| + 2|\beta| + 3\gamma + 6)_q(\alpha_2 + \beta_1 + \beta_2 + \gamma + 2)_q \\
& \times (\alpha_2 + \beta_1 + \beta_3 + \gamma + 2)_q(\alpha_2 + \beta_2 + \beta_3 + \gamma + 2)_q\}.
\end{aligned}
$$

By (10) this equals

$$
\begin{aligned}
H =& \prod_{i=1}^{2}(\alpha_i + |\beta| + 2\gamma + 3)_q \prod_{i=1}^{2}(\alpha_i + |\beta| + \gamma + 3)_q \\
& \times (q^{\alpha_2} - q^{\alpha_1})(|\alpha| + \beta_1 + |\beta| + 2\gamma + 4)_q \\
& \times (|\alpha| + \beta_2 + |\beta| + 2\gamma + 4)_q(|\alpha| + \beta_3 + |\beta| + 2\gamma + 4)_q.
\end{aligned}
$$

Consequently this implies

$$
\begin{aligned}
F(q) =& C_0 q^{|\beta|+2\gamma+6} \frac{(q;q)_{|\alpha|+|\beta|+\gamma+|\delta|+3}}{(q;q)_{|\alpha|+|\beta|+\gamma+|\delta|+f+4}} \frac{(q^{\alpha_2} - q^{\alpha_1})(\alpha_1)_q(\alpha_2)_q(\gamma)_q^2}{(|\alpha| + |\beta| + 2\gamma + 3)_q(|\alpha| + |\beta| + \gamma + 3)_q} \\
& \times \frac{\prod_{j=1}^{3}(|\alpha| + |\beta| + \beta_j + 2\gamma + 4)_q \prod_{1 \le i < j \le 3}(q^{\beta_j} - q^{\beta_i})}{\prod_{i=1}^{2}\prod_{j=1}^{3}(\alpha_i + |\beta| - \beta_j + \gamma + 2)_q \prod_{j=1}^{3}(\beta_j + \gamma + 1)_q \prod_{j=1}^{3}(\beta_j + 1)_q}.
\end{aligned}
$$

Substituting C_0, we have

$$
\begin{aligned}
F(q) =& \frac{q^{\sum_j \binom{\beta_j+2}{2} + \binom{\gamma+3}{2}}}{\prod_{i=1}^{2}(q;q)_{\alpha_i-1}\prod_{j=1}^{3}(q;q)_{\beta_j}(q;q)_{\gamma-1}^2} \frac{(q;q)_{|\alpha|+|\beta|+\gamma+|\delta|+3}}{(q;q)_{|\alpha|+|\beta|+\gamma+|\delta|+f+4}} \\
& \times \frac{(q^{\alpha_2} - q^{\alpha_1})\prod_{1 \le i < j \le 3}(q^{\beta_j} - q^{\beta_i})}{(|\alpha| + |\beta| + 2\gamma + 3)_q(|\alpha| + |\beta| + \gamma + 3)_q} \\
& \times \frac{\prod_{j=1}^{3}(|\alpha| + |\beta| + \beta_j + 2\gamma + 4)_q}{\prod_{i=1}^{2}\prod_{j=1}^{3}(\alpha_i + |\beta| - \beta_j + \gamma + 2)_q \prod_{j=1}^{3}(\beta_j + \gamma + 1)_q \prod_{j=1}^{3}(\beta_j + 1)_q}.
\end{aligned}
$$

This prove the theorem. □

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